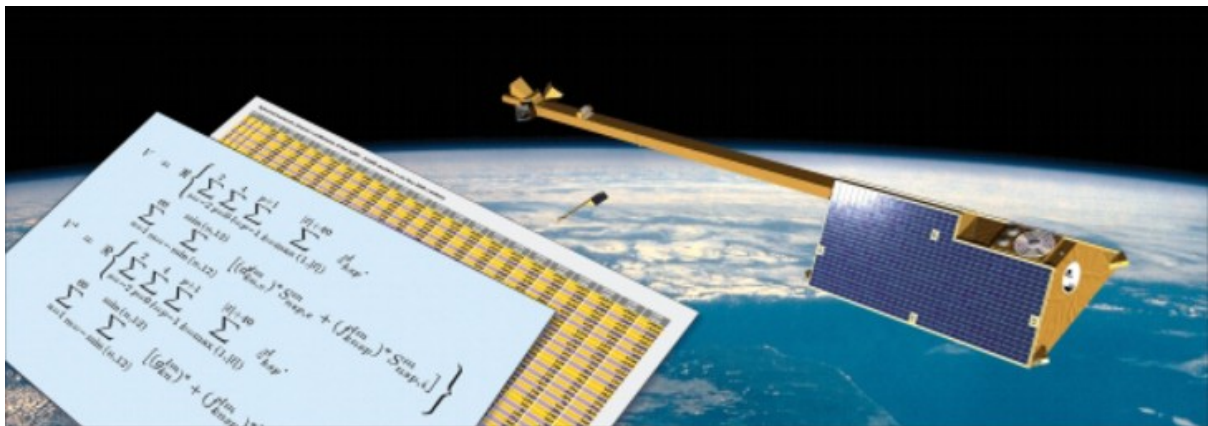




Study of an Improved Comprehensive Magnetic Field Inversion Analysis for *Swarm*



Danish National Space Center (DNSC)

Draft Report

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Preface

This document reports on a part of the activities that have been performed during the first half of the study activities. In its present form it is a “living document” which presents the status of June 21, 2006.

Work presented here has been conducted by Nils Olsen, Terence J. Sabaka and Luis R. Gaya-Pique, with strong support from Alexei Kuvshinov and Lars Tøffner-Clausen.

Chapter 1

Introduction



This report describes the results of activities performed during the Science Study Study “Study of an Improved Comprehensive Magnetic Field Inversion Analysis for *Swarm*” (ESTEC Contract No. 11570/05/NL/AR).

The goal of this activity is to investigate the improvement in modeling Earth’s magnetic field that can be obtained using advanced algorithms for analyzing *Swarm* magnetic field observations, including the differences of the magnetic field measured by the two lower satellites.

The proposed activity is a natural extension of the *Swarm* End-to-End Mission Performance Analysis [Olsen et al., 2004] performed by our team during Phase A of the mission, and the outcome of that activity will be used as a starting point for the current study.

The objectives of this study are:

- An adaptation of the Comprehensive Inversion (CI) approach to account for magnetic vector differences, and to treat the magnetic data in the measurement frame.
- An optimization of the analysis chain to be able to “quickly” respond to questions related to magnetic field mission performance.
- An analysis of the impact of product quality degradation on the mission performance related to geomagnetic field models.

Regarding the first objective, improvement is expected by taking advantage of the constellation aspect through treatment of the lower satellite pair vector data as magnetic field differences, together with an optimal combination with data from the high satellite. This requires a design and implementation of procedures for handling the differences of the vector magnetic data and tuning the comprehensive inversion approach to include differences in an optimal sense. It also includes an in-flight co-estimation of the alignment parameters defining the rotation between the vector magnetometer and the star imager. A mission performance analysis with the most recent choices of orbits and launch dates will be performed including specific “failure” cases. During the development process of improving the End-to-End mission performance simulator, special emphasis will be put on optimizing procedures to arrive at “quick” response tools for assessment of mission performance relevant to models of the internal magnetic field. This will be used during the development and first operational phase of the mission.

1.2 Study Logic

The study has been organized in three main tasks and structured as indicated in Figure 1.1.

The forward scheme for the production of the synthetic data is described in chapter 2.

Impact of the data sampling rate on the recovery of the lithospheric field is discussed in chapter 3.

Lithospheric field recovery using magnetic field differences measured by the lower satellite pair (*Swarm A* and *B*) is the topic of chapter 4.

While all the results obtained in chapters 3 and 4 are obtained using perfectly aligned data (i.e. vector data in the NEC system), a mult-satellite in-flight alignment is discussed in chapter 5.

1.3 Test quantities and criteria

For the present study we used similar assesment criteria as for the Phase A E2E activities. In particular:

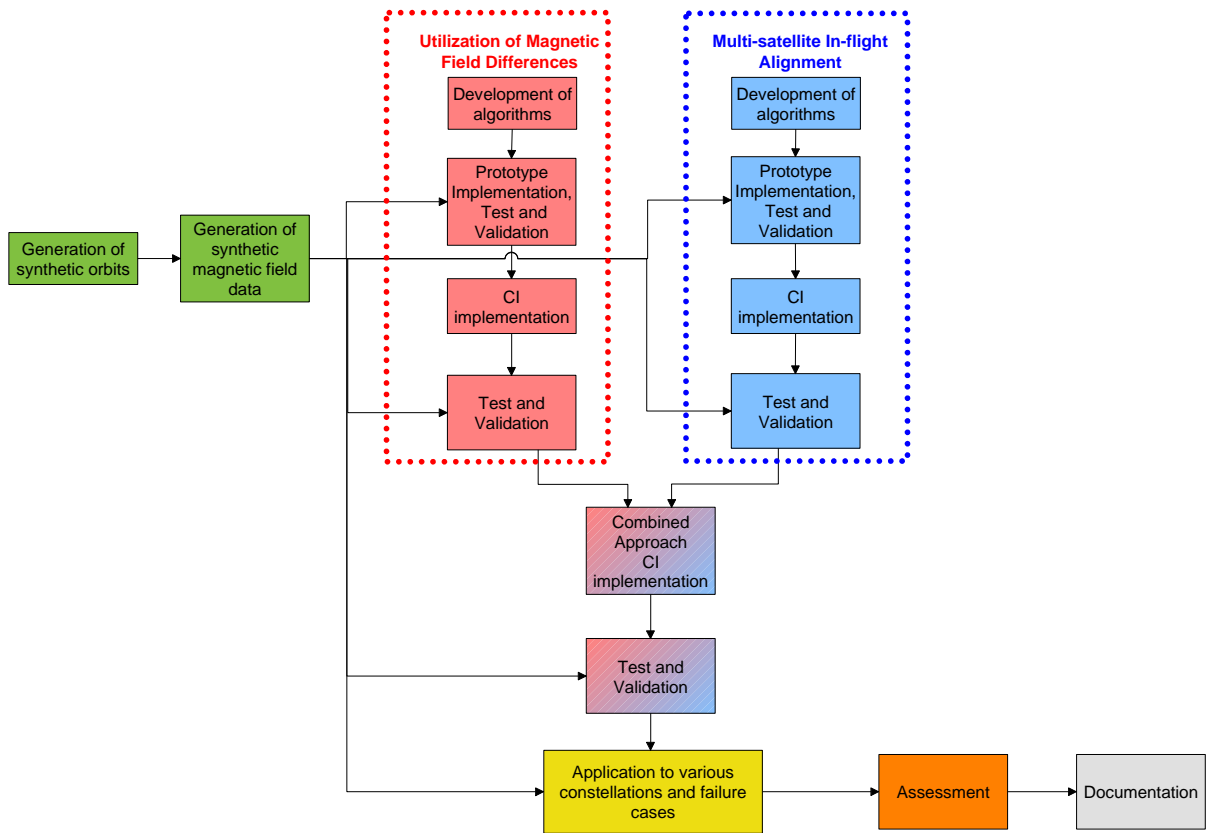


Figure 1.1: Study Logic

- Difference in spectra, degree error, and accumulated error.** The Mauerberger-Lowes spectrum (degree variance), R_n , of the differences between the original and the recovered model coefficients, $\Delta g_n^m, \Delta h_n^m$, in combination with the spectrum of the original model, has been used to evaluate a recovered model. Degree error is defined as $\sqrt{R_n}$, and accumulated error at degree n is defined as $\sqrt{\sum_{l=l_{\min}}^n R_l}$.
- Difference in spectra, azimuthal error.** Although the R_n spectrum provides the mean-squared field magnitude at Earth's surface for degree n , it is rotationally invariant and provides no phase information. Alternatively, one may look at the power spectrum as a function of the ratio $a = m/n$, reminiscent of an azimuthal number, which varies from 0 for zonal terms to 1 for g_n^n and -1 for h_n^n sectorial terms. This spectrum, denoted as R_a , provides the mean-squared field magnitude at Earth's surface for azimuth a . Because the cardinality of the set of a values is usually large, R_a is often smoothed in order to reveal general trends.
- Degree correlation, ρ_n ,** [Langel and Hinze, 1998, eq. 4.23] between the original and the recovered model has also been used to evaluate a recovered model. Models are considered compatible up to that degree n where ρ_n drops below 0.7.
- Sensitivity matrix** is the relative error of each coefficient in a degree versus order matrix and was used to investigate systematic errors. The difference (recovered minus original model) of all the coefficients is determined and subsequently normalized by the mean spectral amplitude of the associated degree n .



- Finally, **global maps** of field differences (for instance of B_r) between the original and the recovered model are used to find geographically confined deficiencies in the recovered models, for instance in connection with the size of the polar gaps.

1.4 Meetings and Workshops

The following meetings and workshops have been held in connection with the activities described in this report:

- **Kick-off meeting (KO)** at DNSC Copenhagen/Denmark, January 27, 2006
- **Progres meeting 1 (PM1)** at DNSC Copenhagen/Denmark, March 29, 2006
- **Mid Term Review (MTR)** at ESTEC Noordwijk/Netherlands, June 26, 2006

Chapter 2

Forward Calculation

This chapter describes the production of synthetic data for the simulated *Swarm* mission. Involved in this task are the working packages WP-1100 "Orbit Determination", and WP-1200 "Magnetic Field Generation".

2.1 Introduction

Figure 2.1 shows a flow chart of the forward scheme.

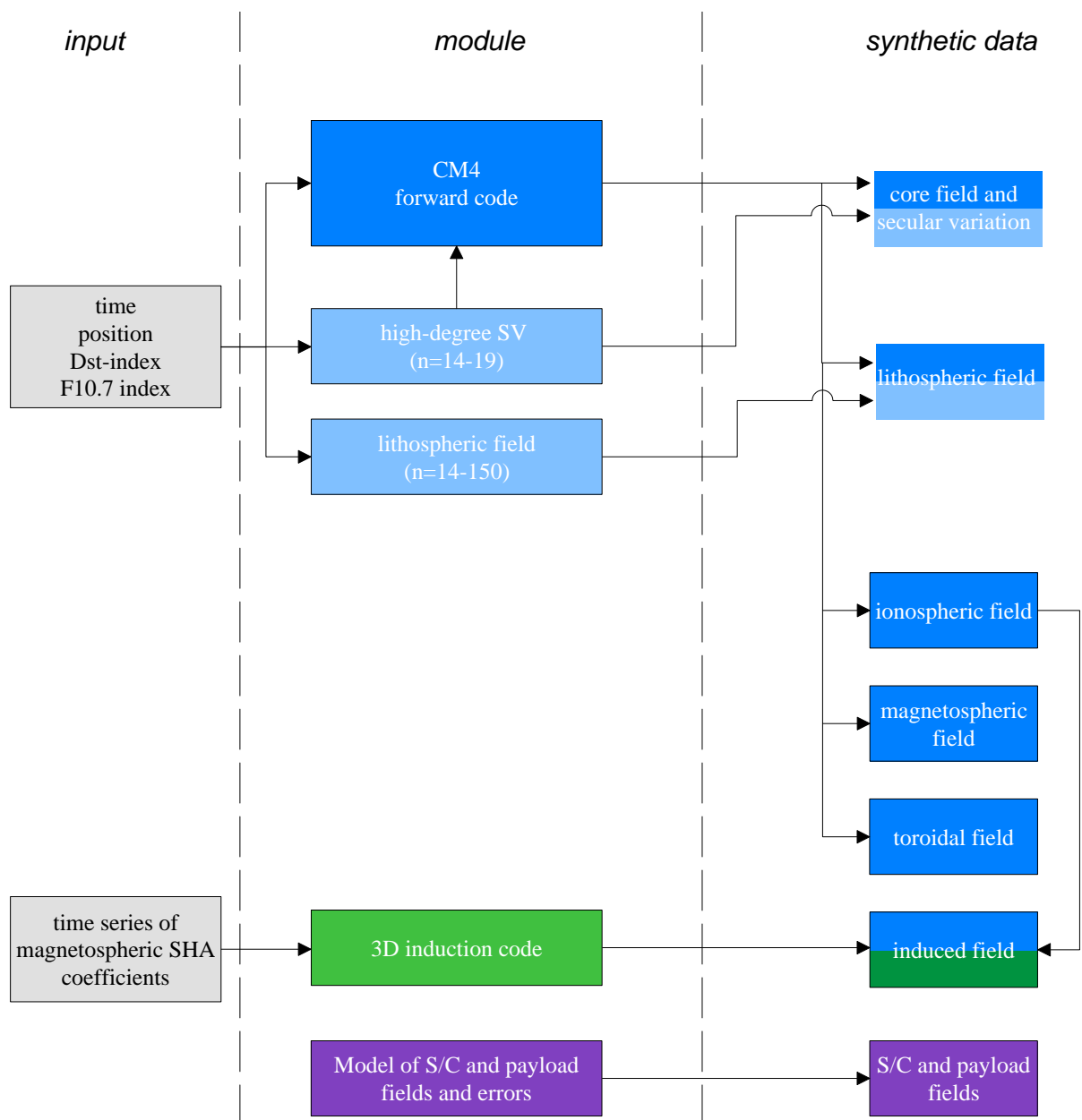


Figure 2.1: Flow chart of the forward scheme.

2.2 A fast approach for generating synthetic orbits

This section describes the approach that we developed for calculating synthetic orbits (positions). This activity relates to WP 1100 “Orbit Determination”.

In the following we only consider the two effects that have the largest impact on orbit evolution

- secular variation in the *Right Ascension of the Ascending Node*, Ω , because of Earth’s oblateness, and
- decay of the orbit altitude due to air drag.

For a circular orbit with semi-major axis a_{sma} and orbit inclination i the nodal drift rate $\dot{\Omega}$ due to Earth’s oblateness (expressed by the dimensionless geopotential coefficient $J_2 = 1082.63 \cdot 10^{-6}$) in the *International Celestial Reference Frame* (ICRF) is given by

$$\dot{\Omega} = -1.5nJ_2 \left(\frac{a}{a_{sma}} \right)^2 \cos(i) \quad (2.1)$$

$$= -2.06474 \cdot 10^{14} a_{sma}^{-7/2} \cos(i) \quad (2.2)$$

[Wertz and Larson, 1999, eq. 6-19], where a is Earth’s (equatorial) radius and n is mean motion. Orbit period T_p is

$$T_p = 2\pi \sqrt{\frac{a_{sma}^3}{\mu}}, \quad (2.3)$$

with $\mu = 3.986005 \cdot 10^{14} \text{m}^3 \text{s}^{-2}$ as the geocentric gravitational constant.

In our approach we first calculate the position vector \mathbf{r}_{ORF} of a circular orbit of radius a_{sma} in a coordinate system that is fixed with the orbit plane (Orbit Reference Frame, ORF). Its z -axis coincides with the z -axis of the *International Celestial Reference Frame* ICRF, but the orbit plane is declined w.r.t. the z axis by the inclination i :

$$\mathbf{r}_{ORF} = \begin{pmatrix} x_{ORF} \\ y_{ORF} \\ z_{ORF} \end{pmatrix} = a_{sma} \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \cos i \\ \sin(\alpha) \sin i \end{pmatrix} \quad (2.4)$$

with

$$\alpha = 2\pi \frac{t - t_0}{T_p} + \nu$$

where t is time, t_0 is epoch, and ν is mean anomaly at epoch t_0 . We then rotate about the z -axis by $\beta = (t - t_0)\dot{\Omega} - \Omega$ to get position in ICRF coordinates:

$$\begin{pmatrix} x_{ICRF} \\ y_{ICRF} \\ z_{ICRF} \end{pmatrix} = \begin{pmatrix} +\cos \beta & -\sin \beta & 0 \\ +\sin \beta & +\cos \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{ORF} \\ y_{ORF} \\ z_{ORF} \end{pmatrix} \quad (2.5)$$

Finally, a rotation about the z -axis by $-GAST$ (where $GAST$ is *Greenwich Apparent Sidereal Time*, cf. Seeber [2004]) yields position and velocity in the coordinates of the *International Terrestrial Reference Frame*, ITRF:

$$\begin{pmatrix} x_{ITRF} \\ y_{ITRF} \\ z_{ITRF} \end{pmatrix} = \begin{pmatrix} +\cos GAST & +\sin GAST & 0 \\ -\sin GAST & +\cos GAST & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{ICRF} \\ y_{ICRF} \\ z_{ICRF} \end{pmatrix} \quad (2.6)$$

After the gravity field the second largest force acting on a satellite at this altitude is the atmospheric drag. The modeling of the atmospheric density is based on the MSIS-86 model [Hedin, 1987] making use of the 3-hourly geomagnetic indices (Ap), as well as the daily solar flux ($F_{10.7}$) and the average solar flux over 3 solar rotations (81 days).

Orbit calculation is done using the following steps:

1. Initial conditions a_{sma} , ν , and Ω are obtained for epoch t_0
2. positions \mathbf{r}_{ITRF} are calculated for one day, i.e. for $t = t_0 - t_0 + 86400$ s and 1 min sampling rate, assuming constant values for a_{sma} and T_p .
3. Air-density ρ is calculated for each location from the MSIS model [Hedin, 1987, 1991], considering the geomagnetic activity Ap and solar radiation $F_{10.7}$
4. The decrease Δa_{sma} of the semi-major axis a_{sma} due to air drag during the day in consideration is calculated in the following way: According to eq. (4.431) of Flury [2000], the change of the semi-major axis *per orbit* is

$$\Delta a_{sma} = -2\pi\bar{\rho}\frac{a_{sma}^2}{B} \quad (2.7)$$

where $\bar{\rho} = \frac{1}{T_p} \int_0^{T_p} \rho dt$ is the mean air density, averaged over one orbit,

$$B = \frac{m}{AC_D} \quad (2.8)$$

is the ballistic coefficient, m is satellite mass, A is the satellite cross-section area, and C_D is the drag coefficient. Since the number of orbits per day is 86400 s/ T_p , the change of the semi-major axis *per day* is

$$\Delta a_{sma} = -\bar{\rho}\frac{\sqrt{\mu a_{sma}}}{B} \quad (2.9)$$

where $\bar{\rho} = \frac{1}{1\text{day}} \int_0^{1\text{day}} \rho dt$ is the mean air density, averaged over one day.

5. The decrease of a_{sma} is linearly interpolated (distributed) over the 24 hours in consideration:

$$\mathbf{r}_{ICRF}^{\text{final}} = \left[1 + (t - t_0) \frac{\Delta a_{sma}}{a_{sma}} \right] \mathbf{r}_{ICRF} \quad (2.10)$$

6. New starting values a_{sma} , ν , and Ω for the new epoch $t_0 = t_0 + 1$ day are calculated and the above steps 2 to 6 are repeated for the next day, until the end of the mission.

2.2.1 Validation of the method

The method is tested by comparison with the observed orbit decay of the CHAMP satellite. For this comparison we used the following initial conditions for the CHAMP orbit: inclination $i = 87.255^\circ$, semi-major axis $a_{sma} = a + 457.1$ km, $\nu = 63.816^\circ$, and $\Omega = 144.43^\circ$ for epoch $t = 213.0$ MD2000 (August 1, 2000, 00:00 UT).

A critical issue is the value of the ballistic coefficient $B = \frac{m}{AC_D}$. A is the effective satellite cross section, which depends on the actual flight direction and on the strength and direction of across-track winds. The cross section area perpendicular to the CHAMP x -axis (which is the nominal flight direction) is $A_x = 0.74$ m²; the value for the two other axes are $A_y = 3.12$ m² and

$A_z = 4.2 \text{ m}^2$. Initial CHAMP satellite mass was $m = 505 \text{ kg}$. Using this mass, A_x as the cross section area, and a value of $C_D = 2.2$ results in a ballistic coefficient of $B = 170 \text{ kg/m}^2$. This value is, however, much lower than those based on observations of the actual CHAMP orbit decay [Liu and Lühr, 2005]. According to Hermann Lühr (personal communication, 2006), an effective cross section area of $A = 0.9 \text{ m}^2$ and a value of $C_D = 2.3 - 2.4$, resulting in $B = 230 \text{ kg/m}^2$ is in much better agreement with the observations. For the following comparison we therefore used $B = 230 \text{ kg/m}^2$.

Figure 2.2 shows the geomagnetic activity index A_p (left) and solar radio flux $F_{10.7}$ (right) for the period in consideration.

Two major orbit maneuvers were performed for CHAMP in 2002, which resulted in an increase of altitude of several kilometers, cf. the black curve in the top panel of Figure 2.3, which presents the observed CHAMP altitude. We accounted for these two orbit maneuvers by increasing the semi-major axis a_{sma} of our prediction by 12.5 km on June 11, 2002, and by 14.5 km on December 10, 2002.

Since the exact value of the ballistic parameter B are not known, we performed the calculations for different values of B . A value of $B = 230 \text{ kg/m}^2$ (blue line) gave the best agreement with observations (black line). The difference between observed and simulated altitude for that values of B is shown in left panel of Figure 2.4; the right panel shows the difference between simulated and observed Local Time.

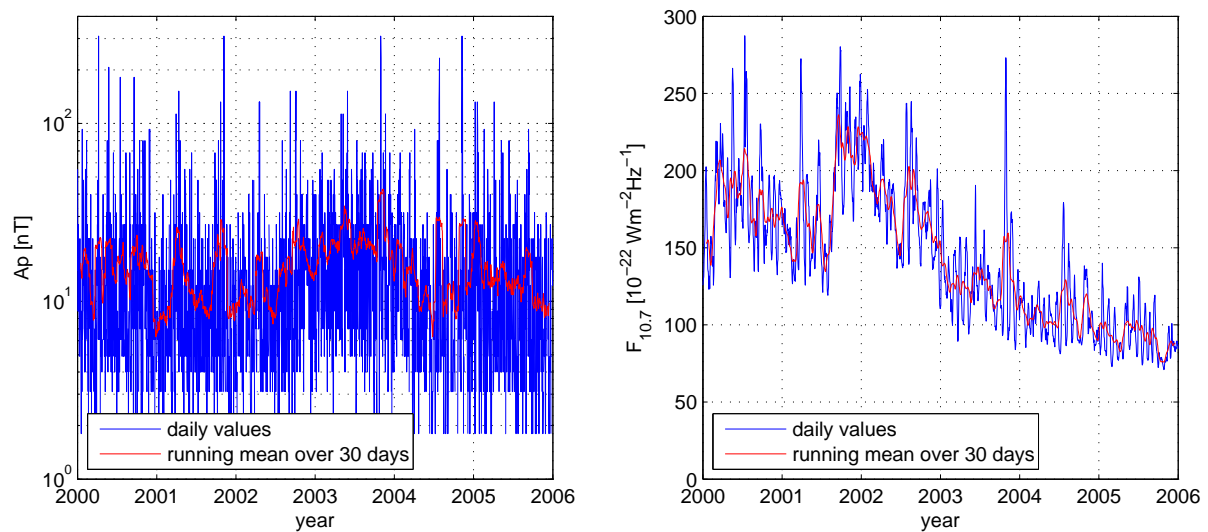


Figure 2.2: Left: Geomagnetic activity index A_p . Right: Solar Radio Flux $F_{10.7}$.

The difference between daily mean values of the simulated and observed CHAMP altitude is less than 3 km, and the difference in Local Time is below 3 min. From these numbers we conclude that our approach is able to predict the main characteristics of a *Swarm*-like satellite orbit.

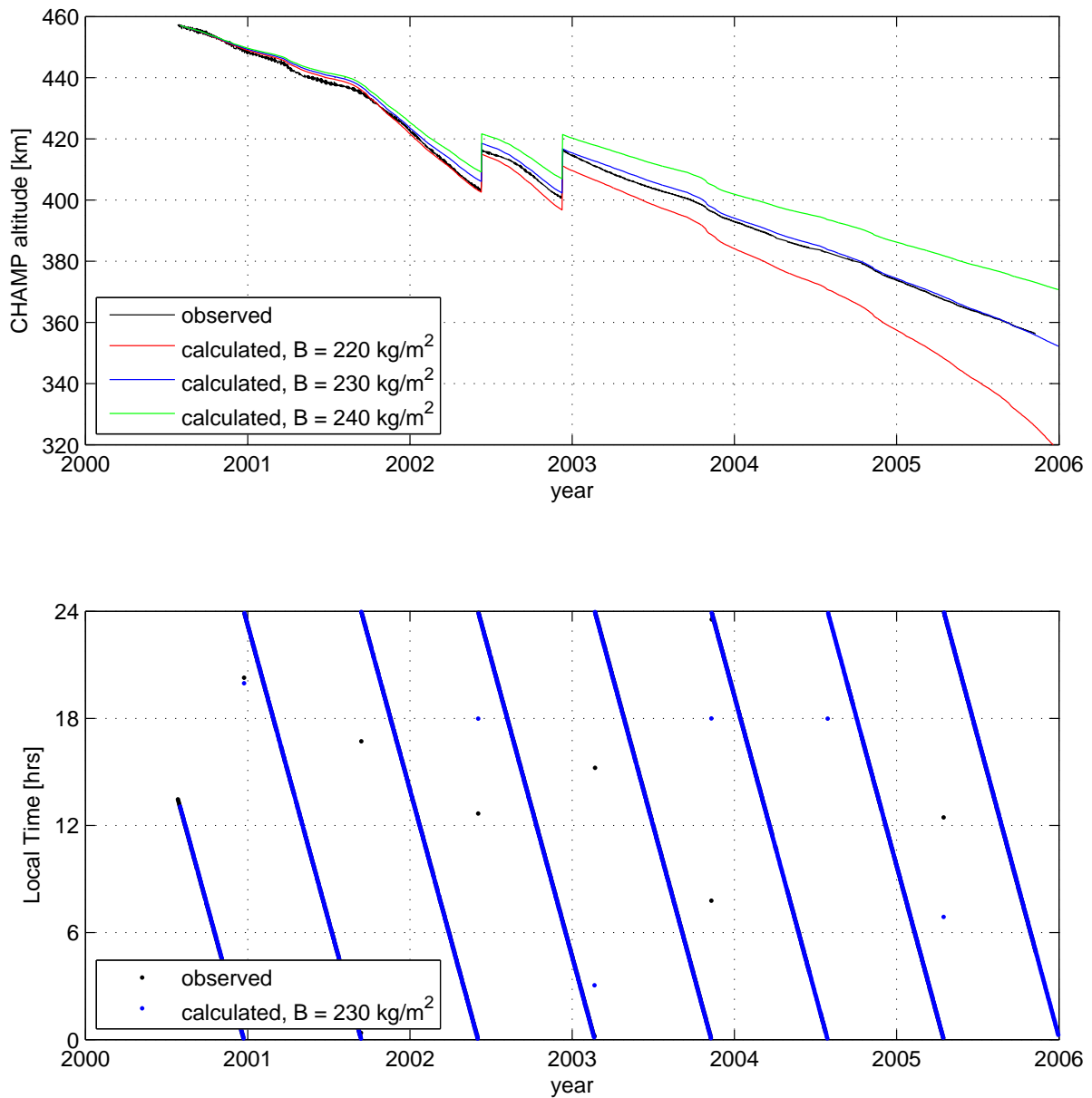


Figure 2.3: Top: observed (black) and simulated orbit decay (daily means) for the CHAMP satellite. Bottom: Observed and simulated Local Time development for CHAMP.

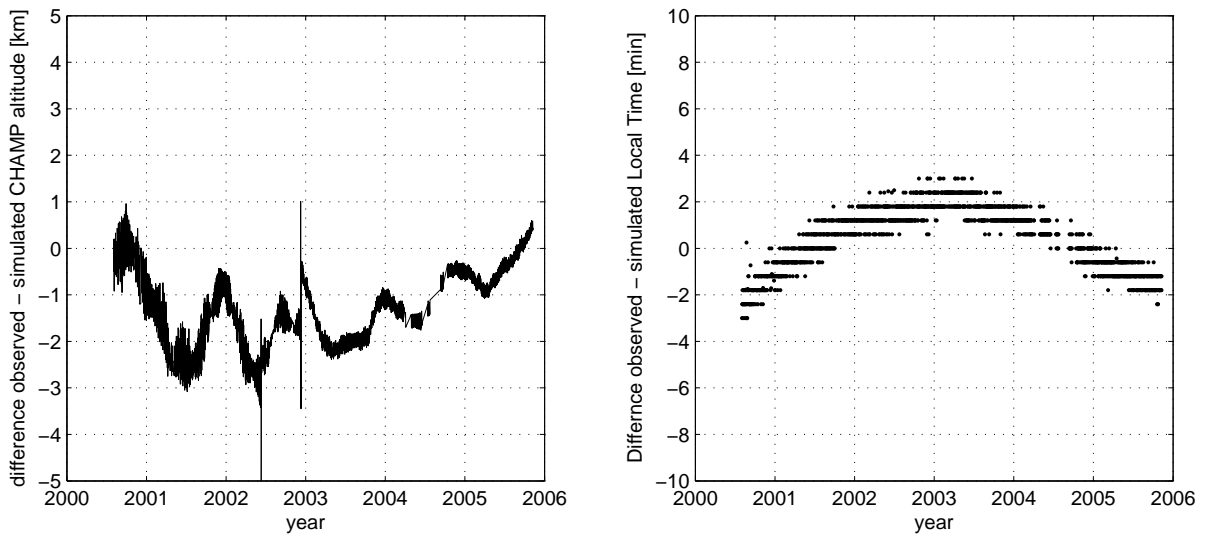


Figure 2.4: Left: Difference between observed and simulated daily means of CHAMP altitude (for $B = 230 \text{ kg/m}^2$). Right: Difference between observed and simulated CHAMP Local Time (for $B = 230 \text{ kg/m}^2$).

2.3 Constellation # 3

For test purposes, we have generated synthetic orbits and data for a constellation – denoted as Constellation #3 (constellation #1 and 2 were the ones analyzed in the E2E mission performance simulation, cf. Olsen et al. [2004]) – that has the following characteristics:

Prerequisites

Epoch for simulations	1997.0 (Launch expected in 2008.0)
Reference radius for altitudes	$a = 6371.2$ km
Earth gravitational constant	$\mu = 3.986004418 \cdot 10^{14}$ m ³ s ⁻² (IERS Conventions 2003)
Drag coefficients	$C_D = 2.2$
Cross-section area	$A = 0.82$ m ⁻² (0.75 m ⁻² + 10% margin)
Satellite mass	$m = 400$ kg

Low Satellites

Swarm A

Inclination	$i = 86.8^\circ$
RAAN	$\Omega = 0^\circ$
Eccentricity	$\epsilon = 0$
Argument of Perigee	$\omega = 0^\circ$
Mean Anomaly at Epoch	$M = 0^\circ$
Mean Altitude	$h \sim 450$ km
Semi-Major Axis	$a_{\text{sma}} = 6821.2$ km

Swarm B

Inclination	$i = 86.8^\circ$
RAAN	$\Omega = 358.5^\circ (= -1.5^\circ)$
Eccentricity	$\epsilon = 0$
Argument of Perigee	$\omega = 0^\circ$
Mean Anomaly at Epoch	$M = 0^\circ$
Mean Altitude	$h \sim 450$ km
Semi-Major Axis	$a_{\text{sma}} = 6821.2$ km

High Satellite

Swarm C

Inclination	$i = 87.3^\circ$
RAAN	$\Omega = 0^\circ$
Eccentricity	$\epsilon = 0$
Argument of Perigee	$\omega = 0^\circ$
Mean Anomaly at Epoch	$M = 90^\circ$
Mean Altitude	$h \sim 530$ km
Semi-Major Axis	$a_{\text{sma}} = 6901.2$ km

Table 2.1: Kepler elements used for Constellation #3, for epoch January 1, 1997, 00:00 UT (launch).

Since this constellation was only used for test purposes, it will not be considered further.

2.4 Constellation # 4

We have used orbit parameters that covers the following points:

- The orbits of all spacecraft shall be circular and near-polar.
- Two satellites (*Swarm A and B*) shall fly at 450 km initial altitude, and separated in longitude by 1.4° .
- The third satellite (*Swarm C*) shall fly at 530 km initial altitude.
- The two orbital planes shall have slightly different inclinations giving rise to a difference in nodal drift by some 90° in 3 years.

Based on this criteria we designed Constellation #4, which has the following characteristics:

For our simulation we assumed a launch of the *Swarm* satellites on July 1, 1998, 00:00 UT, i.e. approximately one solar cycle (11 years) before the anticipated launch in 2010. Start of the simulation one solar cycle before the anticipated launch was done in order to have similar ambient conditions, but have access to actual input values to parameterize e.g. the atmospheric drag model, the Earth rotation variations, etc. And start in July 1998 (rather than January 1999, which would have been exactly 11 years before the anticipated launch in 2010) was done since this terminate the simulated mission before January 2003. This is necessary in order to use the rather sophisticated model of magnetospheric contributions that we developed during Phase A and which is only valid between 1997.0 and 2003.0

Observed and predicted solar flux and geomagnetic activity for the years 1980 - 2015 are shown in Figure 2.5. Note that geomagnetic activity lags solar activity, especially concerning the activity minima.

Due to increasing solar activity, a launch later than July 1998 would require more attitude maintenance in order to guarantee a 4.5 year mission lifetime (assuming a 6 months commissioning phase followed by 4 years of science phase). This is illustrated in Figure 2.6 which shows orbit decay as a function of mission lifetime, for several launch dates and orbit maintenance scenarii.

In order to get a mission of 4.5 years duration in combination with a launch in July 1998, we included four orbit maneuvers for the lower pair: we increased the altitude of *Swarm A* and *B* by 15 km on April 1, 2001 and September 1, 2001, respectively, and by 12 km on November 15, 2001 and April 1, 2002, respectively. The obtained altitude is shown by the blue curve in the left panel of Figure 2.6.

Obtained Local Time, Local Time difference between the upper satellite and the lower pair, and satellite altitude as a function of time is shown in Figure 2.7.

The orbits of *Swarm B* were calculated from those of *Swarm A* in the following way:

- Shift of longitude by 1.4°
- Shift of time stamp by Δt , where Δt increases linearly from 2 secs to 10 secs within 60 days, followed by a re-set to the initial value of 2 secs (saw-tooth function). This simulates the requirement that “The maximum time difference between Swarm A and Swarm B when crossing the equator shall be 10 seconds”.

Prerequisites

Epoch for simulations	1998.5 (Launch expected in 2010)
Reference radius for altitudes	$a = 6371.2 \text{ km}$
Earth gravitational constant	$\mu = 3.986004418 \cdot 10^{14} \text{ m}^3\text{s}^{-2}$ (IERS Conventions 2003)
Ballistic coefficients	$B = 230 \text{ kg/m}^2$

Low Satellites

Swarm A

Inclination	$i = 87.4^\circ$
RAAN	$\Omega = 0^\circ$
Eccentricity	$\epsilon = 0$
Argument of Perigee	$\omega = 0^\circ$
Mean Anomaly at Epoch	$M = 0^\circ$
Mean Altitude	$h \sim 450 \text{ km}$
Semi-Major Axis	$a_{\text{sma}} = 6821.2 \text{ km}$

Swarm B

Inclination	$i = 87.4^\circ$
RAAN	$\Omega = 358.6^\circ (= -1.4^\circ)$
Eccentricity	$\epsilon = 0$
Argument of Perigee	$\omega = 0^\circ$
Mean Anomaly at Epoch	$M = 0^\circ$
Mean Altitude	$h \sim 450 \text{ km}$
Semi-Major Axis	$a_{\text{sma}} = 6821.2 \text{ km}$

High Satellite

Swarm C

Inclination	$i = 88.0^\circ$
RAAN	$\Omega = 0^\circ$
Eccentricity	$\epsilon = 0$
Argument of Perigee	$\omega = 0^\circ$
Mean Anomaly at Epoch	$M = 90^\circ$
Mean Altitude	$h \sim 530 \text{ km}$
Semi-Major Axis	$a_{\text{sma}} = 6901.2 \text{ km}$

Table 2.2: Kepler elements used for Constellation #4, for epoch July 1, 1998, 00:00 UT (launch).

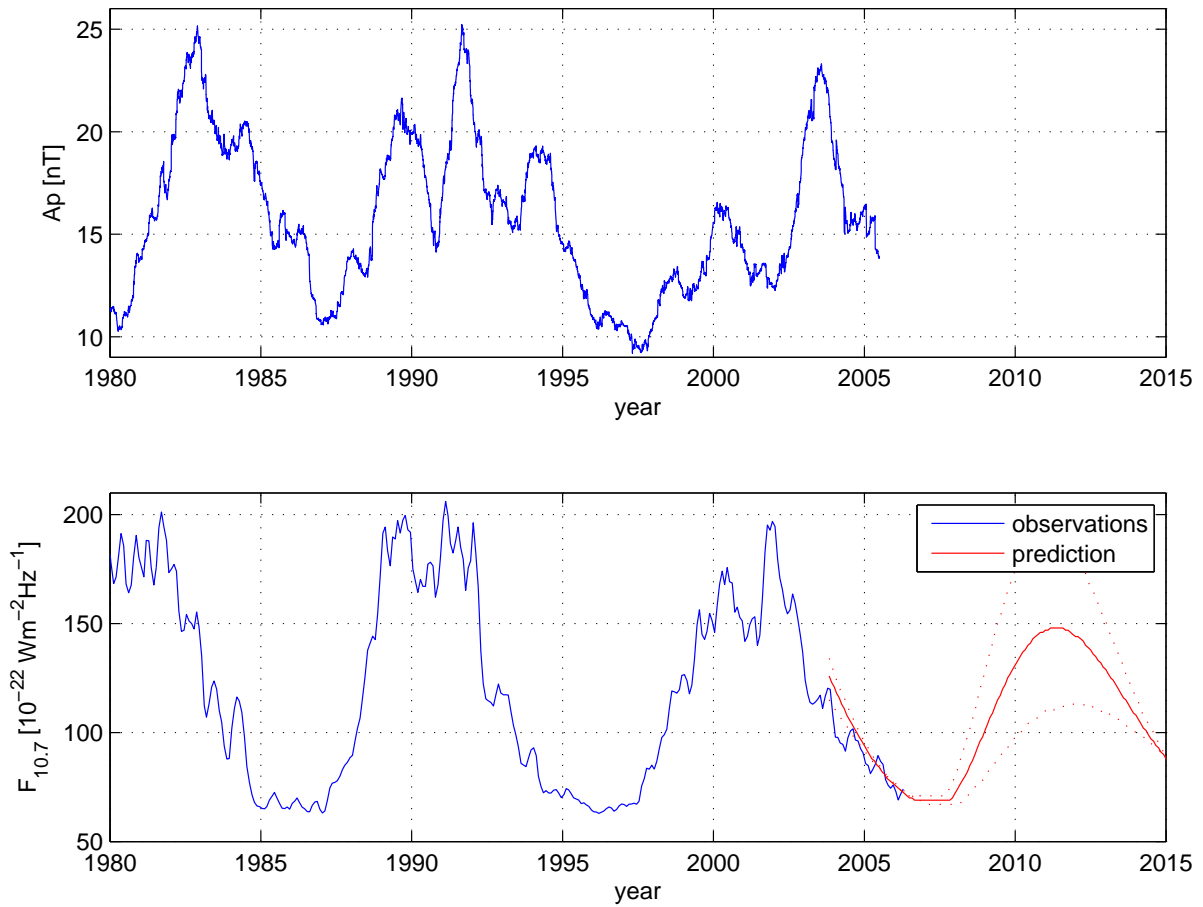


Figure 2.5: Geomagnetic (top) and solar (bottom) activity. Observations are shown in blue, predictions (by K. Schatten) in red.

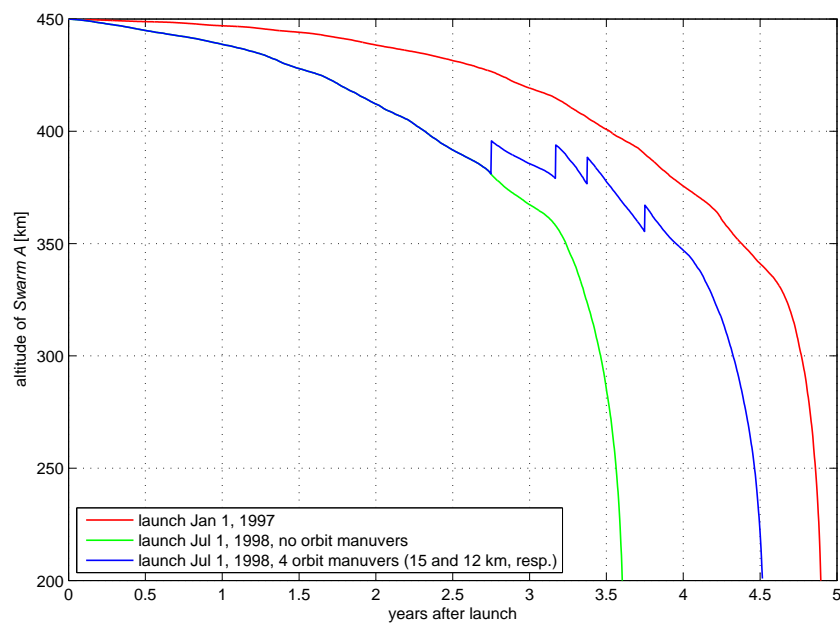


Figure 2.6: Orbit decay for *Swarm A*, for various launch times.

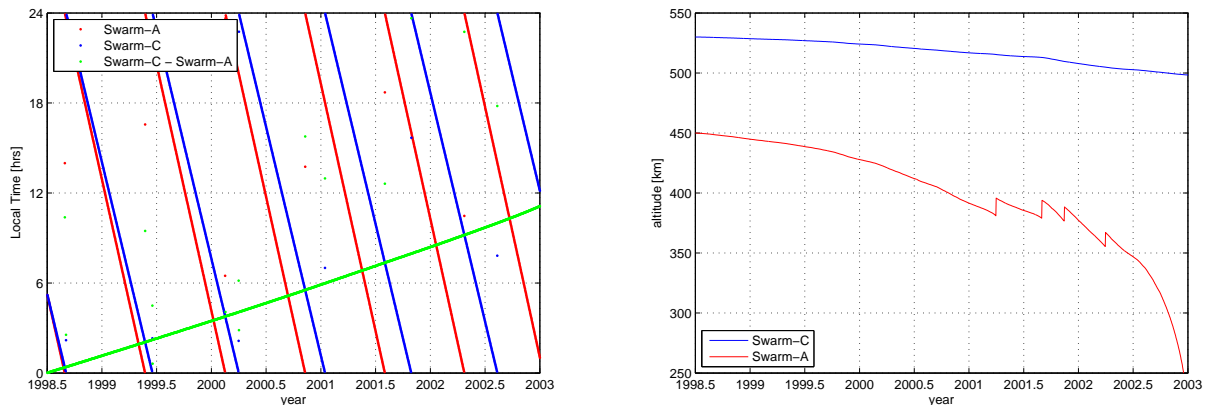


Figure 2.7: Left: Local Time, Local Time difference between the upper satellite and the lower pair, for constellation # 4. Right: obtained altitude.

2.5 Magnetic Field Generation

Magnetic field generation follows mainly the approach used during Phase A of the mission [Olsen et al., 2004, section 3.5]. However, since the toroidal synthetic data generated in Phase A were found to be unrealistic (due to the extrapolation from the Ørsted altitude between 650 and 850 km down to *Swarm* altitudes of 450 km and lower) and thus not included in the Phase A field recovery, we now generated toroidal magnetic fields assuming a fixed altitude of 750 km (i.e. the mean Ørsted altitude). This implies that the in-situ ionospheric currents that are responsible for the toroidal magnetic field do not change with altitude.

Data availability

Data of Constellation #4 are available as daily files with 15 secs sampling rate at ftp://spacecenter.dk/data/magnetic-satellites/Swarm/E2Eplus/constellation_4/. The data are provided as text files (gzipped); The filename follows the naming scheme `swarm4.4.991201.txt.gz` which means data of satellite swarm4, Constellation #4, and for December 1, 1999.

Chapter 3

Impact of the sampling rate: a re-analysis of Phase A data

This section discusses the impact of the sampling rate on the quality of the high-degree lithospheric field recovery. Involved in this task are the working packages WP-3100 “Mission Performance Analysis”, and WP-3200 “Mission Performance Analysis, Support”.

Recall from the Phase A Task 2 and 3 crustal field recoveries that CI performed well for $n < 90$, but that the results of *Maus et al.* were superior for $n \geq 90$. The high degree recovery in CI was hampered by a peculiar contamination that was manifested as a “hemispherical” degradation pattern in the sensitivity matrix centered on $m = 0$ with roughly $n > 84$ shown in Fig. 3.1 along with that of *Maus et al.*. It was unclear as to whether the feature was due to

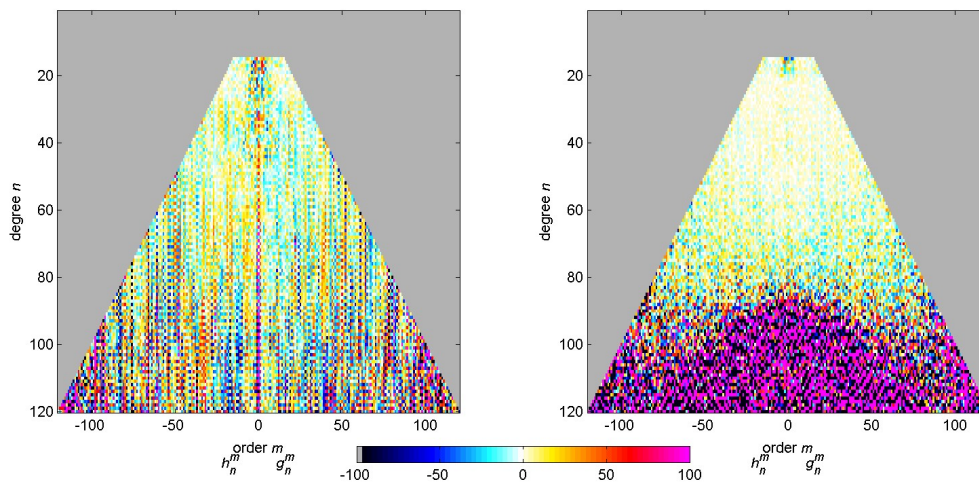


Figure 3.1: Sensitivity matrices for the lithospheric field recovery of two different methods. Left: Gradient approach by *Maus et al.*, cf. section 6.2.8 of the Final E2E Report. Right: Three satellite Comprehensive Inversion result by *Sabaka et al.*, cf. section 6.1.7 of the Final E2E Report.

contamination from a particular magnetic source or perhaps a deficiency in data distribution. Therefore, a series of three test cases were performed over the last year of the mission: only the crustal source was included in the data and parameterized in the model; the magnetospheric field and its parameterization were added; and then finally the field induced by the magnetosphere and its parameterization were added. The results show near perfect recovery when only crust is considered, but the same degradation pattern when adding the magnetosphere and then the induced field. These results indicate that a portion of the time-varying, non-crustal field is being fit by crustal field parameters corresponding consistently to $n > 84$.

Consider that the sampling rate of constellation #1 data is once per minute with an angular speed of the low satellites at approximately $3.84^\circ - 4.03^\circ/\text{min}$. This means that Fourier circular functions having wavenumbers above about $k = 88$ are free to oscillate about a low satellite orbit, which overlaps the transition from $n = 85 - 90$ to the degraded regime. In terms of surface spherical harmonics, let this orbit plane coincide with the equatorial plane of a spherical coordinate system, i.e., the plane $\theta = \pi/2$. In this system the harmonics of order $m > 88$ would be free to oscillate between the sampling nodes. If the orbit plane is now rotated by its inclination angle into the usual geographic system, then the pattern of freely oscillating coefficients would change, but the degree bounds of affected coefficients would remain $n > 88$ since surface spherical harmonics of degree n are an irreducible representation of the rotation group. To illustrate this, spherical harmonic coefficients of $m > 88$ were set to unity, otherwise

zero. The coefficient table is shown in the left panel of Fig. 3.2. These coefficients were then

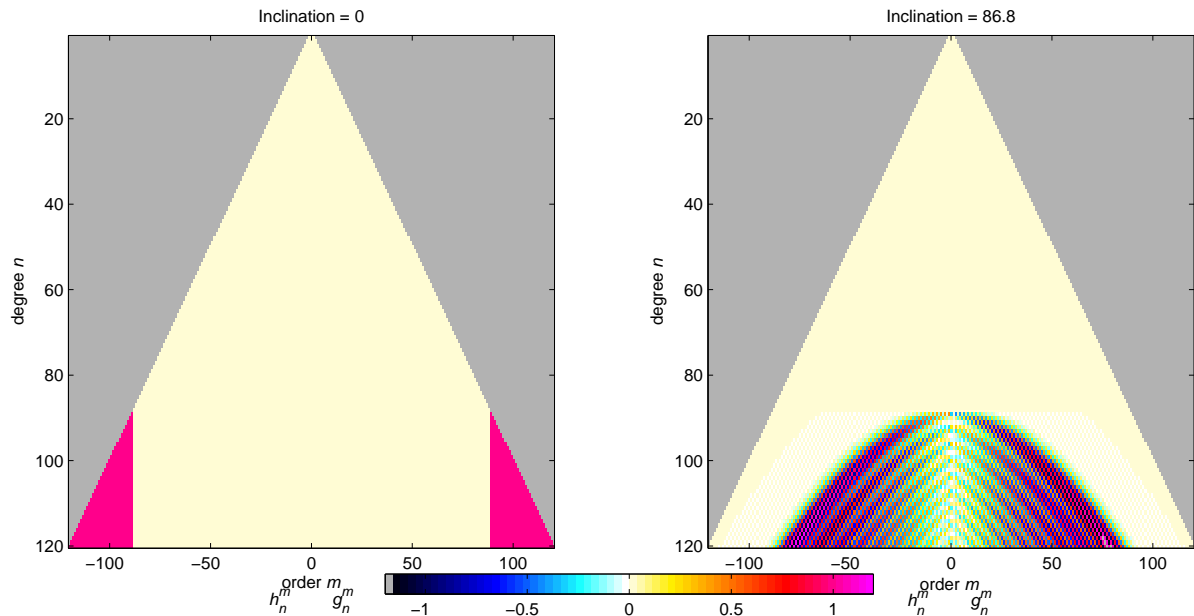


Figure 3.2: Coefficient tables with respect to two different coordinate systems. Left: Coefficients of unity for $m > 88$, otherwise zero, in a reference coordinate system. Right: Same coefficients, but in a system which has been rotated by 86.8° about the cartesian Y axis.

rotated by the low satellite inclination angle of 86.8° about the geographic cartesian Y axis, and the resulting pattern is shown in the right panel of the same figure. This rotation has dramatically changed the pattern such that it looks very similar to the degradation pattern seen in the sensitivity matrix for CI in Fig. 3.1. Although the minimum n of the CI crustal sensitivity matrix is a bit lower ($n = 86$), this strongly suggests that the degradation patterns seen in Phase A Task 2 and 3 may be related to some type of along-track undersampling due to the one minute sampling rate.

The previous discussion is based upon the sampling along a single orbit, but the task results are a function of the total orbit coverage from all satellites over the mission. Previous “crustal only” results clearly show that over time there is sufficient densification of sampling points such that the crustal harmonics are very well determined. Experiments by Sabaka and Olsen [2006] show that the same degradation pattern emerges when dealing with synthetic crustal and magnetospheric signals through either co-estimation or serial estimation. This means that non-crustal signal not fit by non-crustal parameters, on average over the sphere, must alias into the static crustal parameters, if the hypothesis is true. To this end, Task 3 was redone using a 30 sec sampling period so that the corresponding Nyquist wavenumber is about $n = 170$. The results show a dramatic improvement in crustal field recovery, which was synthesized and modelled to $n = 120$, well below the Nyquist wavenumber. The task was further redone using the $n = 150$ crustal model described in the Final E2E Report using both 30 sec and 15 sec sampling periods. The deviation R_n , coefficient difference, and degree correlation plots are shown in Fig. 3.3 for these two cases and the Maus et al. case, the best to date. Clearly the increased along-track sampling rate has made an enormous improvement, even from a 30 to 15 sec period, and has allowed for a satisfactory recovery of crustal field coefficients through the entire $n = 150$ spectrum and not just to $n = 140$ as in Maus et al.. Further information is

provided by sensitivity matrices in Fig. 3.4 where faint spikes are seen near $m = 0$ and $m = \pm 32$ above about $n = 65$ for the CI recoveries, but whose elements are generally much nearer zero than in the *Maus et al.* case.

To understand such a marked improvement, consider that both the induced and magnetospheric fields are linearly interpolated in time between static expansions centered on the hour. This means that it is highly likely that any two consecutive samples at one minute or shorter periods will be highly positively correlated, and so also the portions of their signal not described by the non-crustal model, which could easily be as significant as the true crustal signal above the Nyquist wavenumber. However, samples taken over the same geographical position but at much

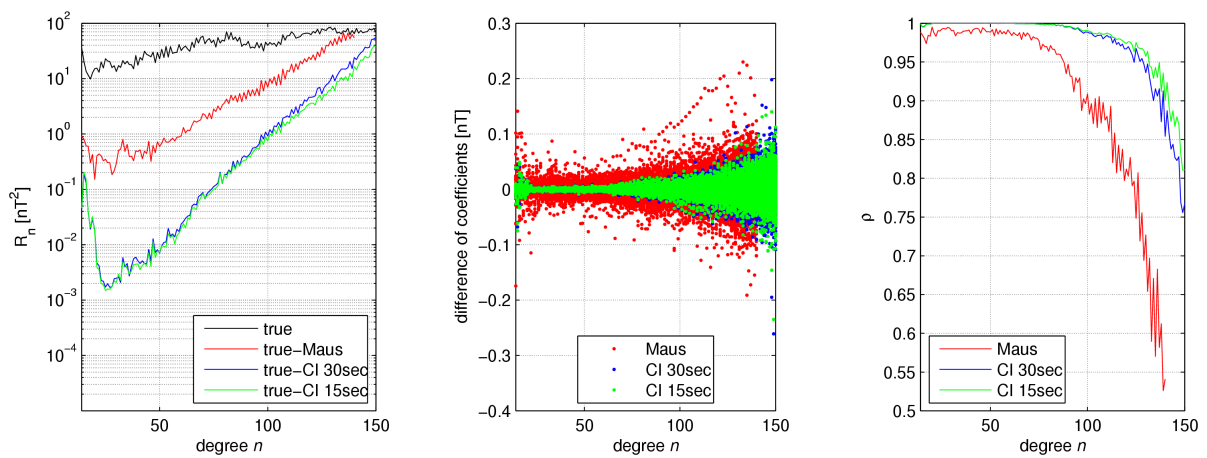


Figure 3.3: Left: R_n spectra of the true (black) crustal field for $n > 13$ and those of its difference with the *Maus et al.* (red), CI 30 sec (blue), and 15 sec (green) sampling period models. Middle: Coefficient differences of the various models (same color scheme) with respect to the true model. Right: Degree correlation of the various models (same color scheme) with respect to the true model.

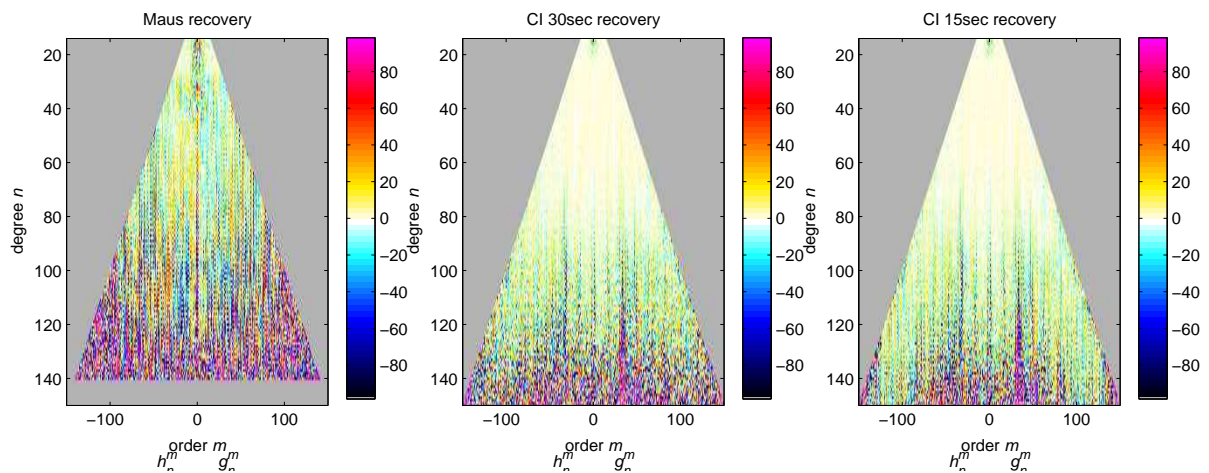


Figure 3.4: Sensitivity matrices for the lithospheric field recovery of two different methods. Left: Gradient approach by *Maus et al.*, cf. section 6.2.8 of the Final E2E Report. Middle: Three satellite Comprehensive Inversion result using a 30 sec sampling period. Right: Same as middle, but with a 15 sec sampling period.

different times will likely not be highly correlated in these remaining signals. Therefore, the leakage of non-crustal signal into crustal parameters is likely driven by consecutive, temporally correlated samples. If these signals are large enough, then the reduction of the least-squares cost function will be worth the crustal field fitting these signals with coefficients above the Nyquist wavenumber. Now if samples are taken every 30 seconds, then there will be a point midway between the original one minute samples. If this point has remaining non-crustal signal which is not positively correlated with that of the two adjacent samples, e.g., it is strongly negative while the adjacent signals are strongly positive, then there is a chance that these three signals can be fit by crustal parameters above the original Nyquist wavenumber. However, if the signal is positively correlated, e.g., all signals are strongly positive or negative, then their fitting will be prohibited since the crustal harmonics above the Nyquist wavenumber must oscillate about the zero level between the one minute samples. That is, a good fit to the one minute samples will result in a large misfit to the 30 sec sample midway between and perhaps an increase in the cost function. It is expected that 30 sec samples will indeed be strongly positively correlated, and so this is thought to be the reason for the large reduction in non-crustal leakage into the crustal spectrum above the Nyquist wavenumber. Note that by this argument the 15 sec period would not be expected to have much advantage over the 30 sec case, and this is corroborated in Figs. 3.3 and 3.4.

In summary, undersampling of the time varying non-crustal field along track, with respect to the crustal field harmonics, can result in aliasing of prominent portions of this field into crustal field parameters. A higher sampling rate can reduce this aliasing, but this probably depends upon the nature of the remaining non-crustal field; if it is not strongly positively correlated from point to point, then this problem may remain. In general, non-crustal models of higher resolution, when supported by adequate data coverage, are the key to mitigating this problem.

Chapter 4

Lithospheric Field Recovery Using Magnetic Field Differences

This chapter describes the development, test and implementation of an approach to utilize the magnetic field difference (“gradient”) of the lower *Swarm* satellite pair. Involved in this task are the working packages WP-2110 “Difference Approach: Algorithm Development”, and WP-2120 “Difference Approach: CI Implementation”.

Perhaps one of the most innovative features of the *Swarm* mission is the east-west pairing of the lower satellites whose near-polar orbit planes are separated by 1.5° . A simple differencing of the data will result in an enhancement of their potential field signal as a function of spherical harmonic order m . Advantage may be taken of this enhancement to resolve crustal structure much smaller than possible by an analysis of the usual field components at the same altitude. These ideal gain factors are shown in Fig. 4.1 for longitudinal separations of $\Delta\phi = 1^\circ$, 1.5° , and 2° .

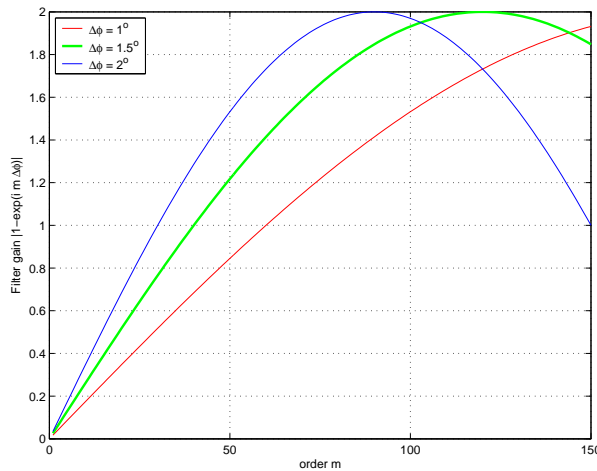


Figure 4.1: Gain factor for order m from data differences at longitude differences of $\Delta\phi = 1^\circ$, 1.5° , and 2° .

A sequential analysis may be undertaken in which non-crustal fields are first removed from either the field components or directly from the difference or gradient data. These data are then fit with a crustal field. However, these types of sequential estimations have been shown to be sub-optimal in their separation of signals from the various sources [Sabaka and Olsen, 2006]. The CI approach attempts to parameterize and estimate all major sources simultaneously to avoid these separation problems. The following section will introduce a methodology in which the sensitivity of the gradient data to small-scale crustal structure is exploited while non-crustal parameters are determined by the whole data set all within the CI framework.

4.1 Selective Infinite-Variance Weighting

The key to this approach is producing and/or identifying data subsets that are particularly sensitive to certain parameter subsets and applying appropriate weighting such that these data strongly influence the determination of such parameters. The weighting employed here will be termed “selective infinite-variance weighting” (SIVW) for reasons now to be discussed. Consider the following simple linear model where the data vector \mathbf{d} is related to the model parameter vector \mathbf{x} through a linear operator A

$$\mathbf{d} = A\mathbf{x} + \nu, \tag{4.1}$$

$$= A\mathbf{x} + B\mathbf{y} + \mathbf{e}, \tag{4.2}$$

where ν is the noise vector having both a random part, \mathbf{e} and a systematic part $B\mathbf{y}$. Here B is a linear operator and \mathbf{y} is the underlying state. Let \mathbf{e} and \mathbf{y} be uncorrelated and normally distributed such that $\mathbf{e} = N(\mathbf{0}, C)$ and $\mathbf{y} = N(\mu, Q)$. Therefore, ν is normally distributed such that $\nu = N(B\mu, C + BQB^T)$. Given that least-squares is the estimator of choice, a critical restriction is then imposed on the treatment of the noise vector, i.e., that which is accounted for in the data covariance matrix W^{-1} , which is its expected value is to be zero. This is equivalent to choosing finite C and Q covariance matrices such that

$$W = (C + BQB^T)^{-1}, \quad (4.3)$$

$$= C^{-1} - C^{-1}B(Q^{-1} + B^T C^{-1}B)^{-1}B^T C^{-1}. \quad (4.4)$$

The corresponding least-squares solution $\tilde{\mathbf{x}}$ is then given by

$$\tilde{\mathbf{x}} = (A^T W A)^{-1} A^T W d, \quad (4.5)$$

$$= (A^T W A)^{-1} A^T W (A\mathbf{x} + B\mathbf{y} + \mathbf{e}), \quad (4.6)$$

$$= \mathbf{x} + (A^T W A)^{-1} A^T W (B\mathbf{y} + \mathbf{e}), \quad (4.7)$$

with expected value

$$E[\tilde{\mathbf{x}}] = \mathbf{x} + (A^T W A)^{-1} A^T W B\mu. \quad (4.8)$$

If $E[\nu] = \mu = \mathbf{0}$, in accordance with least-squares, then the solution is unbiased, otherwise not. This violation in the assumption of zero expected value of the noise occurs frequently in geomagnetic field modelling, usually in the context of unparameterized natural field sources. Because they are accounted for solely through W^{-1} by attempting to assign a finite variance, they are necessarily assumed to be zero-mean by the restrictions imposed by least-squares. Note that even if a natural field is a zero-mean process over some characteristic time interval, it most certainly does not have a zero-valued realization at the time a measurement was taken.

Aside from the restriction of zero-mean noise processes, one is free to choose the variance, σ^2 , of the process in the least-squares weighting. Recall that as σ^2 increases, the distribution curve remains centered on zero, but becomes flatter. In fact, for finite σ^2 , the distribution is always centered on zero. However, for infinite variances the central tendency, and thus, the mean of the distribution becomes undefined. This is a very useful result in that it allows for undefined (non-zero) means within the restrictive framework of least-squares. Now let $Q = \sigma^2 \tilde{Q}$ and let $\sigma^2 \rightarrow \infty$ in eq. 4.4 such that

$$W_\infty \equiv \lim_{\sigma^2 \rightarrow \infty} W, \quad (4.9)$$

$$= \lim_{\sigma^{-2} \rightarrow 0} C^{-1} - C^{-1}B(\sigma^{-2}\tilde{Q}^{-1} + B^T C^{-1}B)^{-1}B^T C^{-1}, \quad (4.10)$$

$$= C^{-1} - C^{-1}B(B^T C^{-1}B)^{-1}B^T C^{-1}. \quad (4.11)$$

The corresponding least-squares solution is now given by

$$\tilde{\mathbf{x}} = (A^T W_\infty A)^{-1} A^T W_\infty d, \quad (4.12)$$

$$= (A^T W_\infty A)^{-1} A^T W_\infty (A\mathbf{x} + B\mathbf{y} + \mathbf{e}), \quad (4.13)$$

$$= \mathbf{x} + (A^T W_\infty A)^{-1} A^T W_\infty \mathbf{e}. \quad (4.14)$$

The term in B in eq. 4.13 has dropped out of eq. 4.14 because

$$W_\infty B = (C^{-1} - C^{-1}B(B^T C^{-1}B)^{-1}B^T C^{-1})B, \quad (4.15)$$

$$= C^{-1}B - C^{-1}B(B^T C^{-1}B)^{-1}B^T C^{-1}B, \quad (4.16)$$

$$= C^{-1}B - C^{-1}B, \quad (4.17)$$

$$= 0. \quad (4.18)$$

In fact, if eq. 4.1 is prewhitened such that $C = I$, then $W_\infty = \mathcal{N}_B$, where \mathcal{N}_B is the projection matrix onto the nullspace of the column span of B . Thus, W_∞ imposes a harder bound on $\tilde{\mathbf{x}}$ than does W such that all signal that can be represented by $B\mathbf{y}$ is eliminated from the estimation. This can be seen from

$$E[\tilde{\mathbf{x}}] = \mathbf{x}, \quad (4.19)$$

and so the solution is always unbiased independent of the state of \mathbf{y} . Sabaka and Olsen [2006] have shown that the solution given in eq. 4.12 is equivalent to the solution one gets when least-squares co-estimating \mathbf{y} with \mathbf{x} in eq. 4.2. Thus, one finds that co-estimating \mathbf{y} with \mathbf{x} is equivalent to treating \mathbf{y} as a normally distributed infinite-variance process requiring infinite-variance weighting.

One can now begin to see the mechanism by which infinite-variance weighting may be used to selectively minimized the influence of certain data subsets on certain parameter subsets. Define two data subsets, \mathbf{d}_1 and \mathbf{d}_2 , which are linear functions of two parameter subsets of interest, \mathbf{x} and \mathbf{y} , associated with linear operators A and B , respectively. Let the corresponding data noise vectors be $\nu_1 = N(\mathbf{0}, I)$ and $\nu_2 = \mathbf{e} + B_2\mathbf{z}$, where $\mathbf{e} = N(\mathbf{0}, I)$ and \mathbf{z} is deterministic and represents a systematic contamination of \mathbf{y} in the \mathbf{d}_2 data. In order to account for ν_1 , \mathbf{e} , and \mathbf{z} the following W is constructed which treats \mathbf{z} as an infinite-variance process

$$W = \begin{pmatrix} I & 0 \\ 0 & I - B_2(B_2^T B_2)^{-1} B_2^T \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & \mathcal{N}_{B_2} \end{pmatrix}. \quad (4.20)$$

Since W is idempotent it may be used directly to prewhiten the augmented system equations

$$\begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} + \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}, \quad (4.21)$$

such that

$$\begin{pmatrix} \mathbf{d}_1 \\ \mathcal{N}_{B_2} \mathbf{d}_2 \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ \mathcal{N}_{B_2} A_2 & \mathcal{N}_{B_2} B_2 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} + \begin{pmatrix} \nu_1 \\ \mathcal{N}_{B_2} \nu_2 \end{pmatrix}, \quad (4.22)$$

$$= \begin{pmatrix} A_1 & B_1 \\ \mathcal{N}_{B_2} A_2 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} + \begin{pmatrix} \nu_1 \\ \mathcal{N}_{B_2} \mathbf{e} \end{pmatrix}. \quad (4.23)$$

Note from eq. 4.23 that this W has selectively eliminated the direct influence of \mathbf{d}_2 on the determination of \mathbf{y} via the B_2 matrix while allowing all data to continue to influence \mathbf{x} . The equivalent system of equations for co-estimation is given by

$$\begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{pmatrix} = \begin{pmatrix} A_1 & B_1 & 0 \\ A_2 & 0 & B_2 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{n} \end{pmatrix} + \begin{pmatrix} \nu_1 \\ \mathbf{e} \end{pmatrix}, \quad (4.24)$$

where $\mathbf{n} = \mathbf{y} + \mathbf{z}$ are considered “nuisance parameters”. From an operational standpoint, solving eq. 4.24 is much preferred to solving eq. 4.23, since the latter requires an explicit formation of the dense matrix \mathcal{N}_{B_2} , which has square dimensions of order $\dim \mathbf{d}_2$.

The application of selective infinite-variance weighting to the current problem is now straight forward. Let \mathbf{d}_1 , \mathbf{d}_2 , and \mathbf{d}_3 be the data from *Swarm* low satellites 1 and 2 and high satellite 3. Assume that $\dim \mathbf{d}_1 = \dim \mathbf{d}_2$ and that their elements are chronologically matched. Let \mathbf{x} be the non-crustal parameters related to the data through A and let \mathbf{y} be the crustal field

parameters related through B . Let ν_1 , ν_2 , and ν_3 be the noise vectors corresponding to the data. Consider the following rotation of the prewhitened system equations

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \mathbf{d}_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \left[\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \\ A_3 & B_3 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} + \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \right] \quad (4.25)$$

$$\begin{pmatrix} \mathbf{d}_s \\ \mathbf{d}_d \\ \mathbf{d}_3 \end{pmatrix} = \begin{pmatrix} A_s & B_s \\ A_d & B_d \\ A_3 & B_3 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} + \begin{pmatrix} \nu_s \\ \nu_d \\ \nu_3 \end{pmatrix}, \quad (4.26)$$

where the subscripted “s” and “d” indicate sums and differences, respectively.

The gain factors in Fig. 4.1 correspond to the difference data only. It turns out that the gain factors for sums of data are complementary to this. That is, if the difference gain for order m is $g_d(m)$, then the corresponding sum gain is $g_s(m) = 2 - g_d(m)$. This suggests that higher order crustal coefficients should be determined by difference data only while the lower order coefficients should be determined by sum data and possibly data from the high satellite. Let the lower order crustal coefficients be designated by \mathbf{y}_ℓ and the higher order by \mathbf{y}_h and let them be related to the data through B^ℓ and B^h , respectively. The resulting system equations are given by

$$\begin{pmatrix} \mathbf{d}_s \\ \mathbf{d}_d \\ \mathbf{d}_3 \end{pmatrix} = \begin{pmatrix} A_s & B_s^\ell & 0 & 0 & B_s^h \\ A_d & 0 & B_d^h & B_d^\ell & 0 \\ A_3 & B_3^\ell & 0 & 0 & B_3^h \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y}_\ell \\ \mathbf{y}_h \\ \mathbf{n}_\ell \\ \mathbf{n}_h \end{pmatrix} + \begin{pmatrix} \nu_s \\ \nu_d \\ \nu_3 \end{pmatrix}, \quad (4.27)$$

where \mathbf{n}_ℓ and \mathbf{n}_h are the nuisance parameters in the low and high order crustal coefficient regimes, respectively. These parameters are a combination of true crustal signal and a systematic noise component due to such things as non-crustal signal leakage. Thus, while the \mathbf{x} parameters are influenced directly by all data, the \mathbf{y} parameters are determined by different combinations of data, depending on their sensitivity. Note that if the distinction between \mathbf{y} and \mathbf{n} were not made, then eq. 4.27 would reduce to the original, unrotated system in eq. 4.25. Clearly, this method may be further refined, for example, by distinguishing again between sums, differences, and high satellite data collected on the dayside versus nightside.

4.2 Application to the Gradient problem

The application of SIVW to the gradient problem was carried out on synthesized data from constellation # 3 using a 15 sec sampling interval. Specifically, the difference data were focused on coefficients of order $m > 20$, while all data, including sums and differences, were focused on $m \leq 20$. The change in recovery of low-order coefficients when difference data are focused on them or not is negligible in this case. Therefore, eq. 4.27 now becomes

$$\begin{pmatrix} \mathbf{d}_s \\ \mathbf{d}_d \\ \mathbf{d}_3 \end{pmatrix} = \begin{pmatrix} A_s & B_s^\ell & 0 & B_s^h \\ A_d & B_d^\ell & B_d^h & 0 \\ A_3 & B_3^\ell & 0 & B_3^h \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y}_\ell \\ \mathbf{y}_h \\ \mathbf{n}_h \end{pmatrix} + \begin{pmatrix} \mathbf{e}_s \\ \mathbf{e}_d \\ \mathbf{e}_3 \end{pmatrix}. \quad (4.28)$$

Fields whose parameters reside in \mathbf{x} include core and associated SV, ionosphere and associated induced, and magnetosphere and associated induced in the form of explicit high-frequency

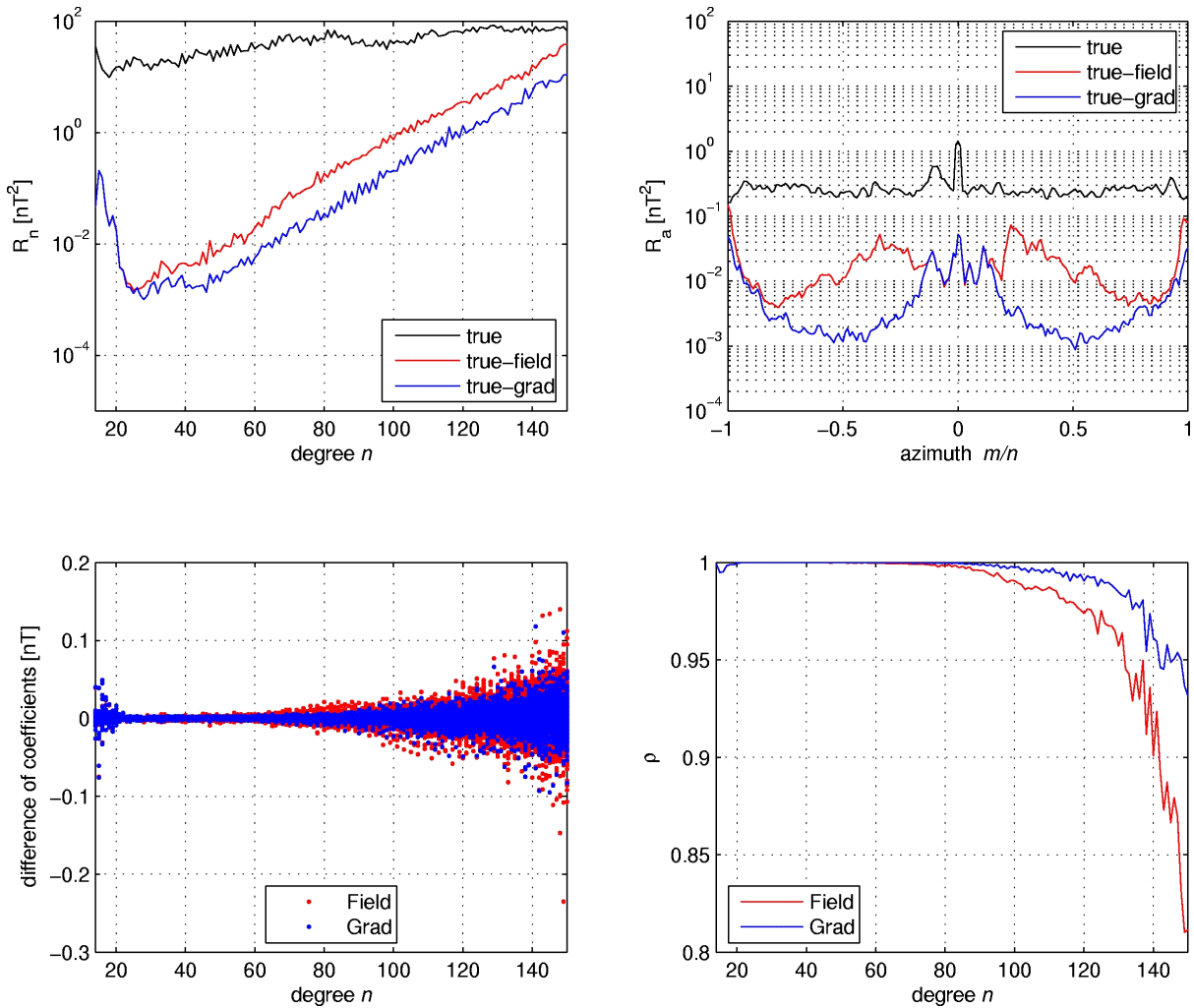


Figure 4.2: Upper-left: R_n spectra of the true (black) crustal field for $n > 13$ and those of its difference with CI applied directly to field values (red) and to low satellite sums and differences and high satellite data (blue). Upper-right: R_a (azimuthal) spectra. Lower-left: Coefficient differences with respect to the true model. Lower-right: Degree correlation with respect to the true model.

internal expansions, and follow the E2E study of Sabaka and Olsen [2006]. The data are prewhitened by the usual equal-area factor, $\sqrt{\sin \theta}$, where θ is the geographic colatitude of the data position.

A comparison of the performance of CI applied directly to the field components measured by the satellites versus a solution to the system in eq. 4.28 is illustrated in Fig. 4.2. The upper-left panel shows the R_n spectra of the former (red), the latter (blue), and the true model (black) and indicates a roughly two-fold reduction in power in the error per degree above about $n = 45$ when difference data are exploited. Looking at the upper-right panel shows an error that is as much as three times less for azimuthal, (m/n), numbers near 0.5, but consistently less when using difference data for $|m/n| > 0.2$. Interestingly, both cases show nearly identical behavior for low azimuthal numbers, which are dominated by low-order m and high-degree n . Recall that this regime ($m \leq 20$) is sensitive to all data, and so this is not unexpected. The lower-left panel shows the coefficient differences between the true and recovered models and confirms that

the SIVW applied to the rotated data system extracts coefficients at least as accurate as the traditional approach, but much more so for high n . The increased spread of the differences for $n \leq 20$ is due to the fact that SV is present and is more of a challenge to resolve than static coefficients. Finally, the lower-right panel indicates a far superior recovery of coefficients with the new method with regards to phase of the models, i.e., degree correlation. The new method is now recovering a field that is positively correlated with the true field at the 0.93 level for $n = 150$, the degree limit of the synthetic signal. Again, note the slight excursion below $n = 20$ due to SV.

A more complete view of the performance of the traditional field value method versus the gradient method can be seen by comparing the sensitivity matrices in Fig. 4.3. The gradient recovery is excellent for mid-valued m , as confirmed by the R_a plots in Fig. 4.2, but is nearly identical to the traditional approach for the $m \leq 20$ regime. There is also some improvement for the purely sectorial terms (left and right edges). All of these results strongly suggest that the *Swarm* constellation is capable of accurately recovering very small-scale lithospheric structure, perhaps even smaller than indicated in these tests. This will go far in closing the gap in intermediate lithospheric wavelength coverage that exists now between satellites and aeromagnetic surveys.

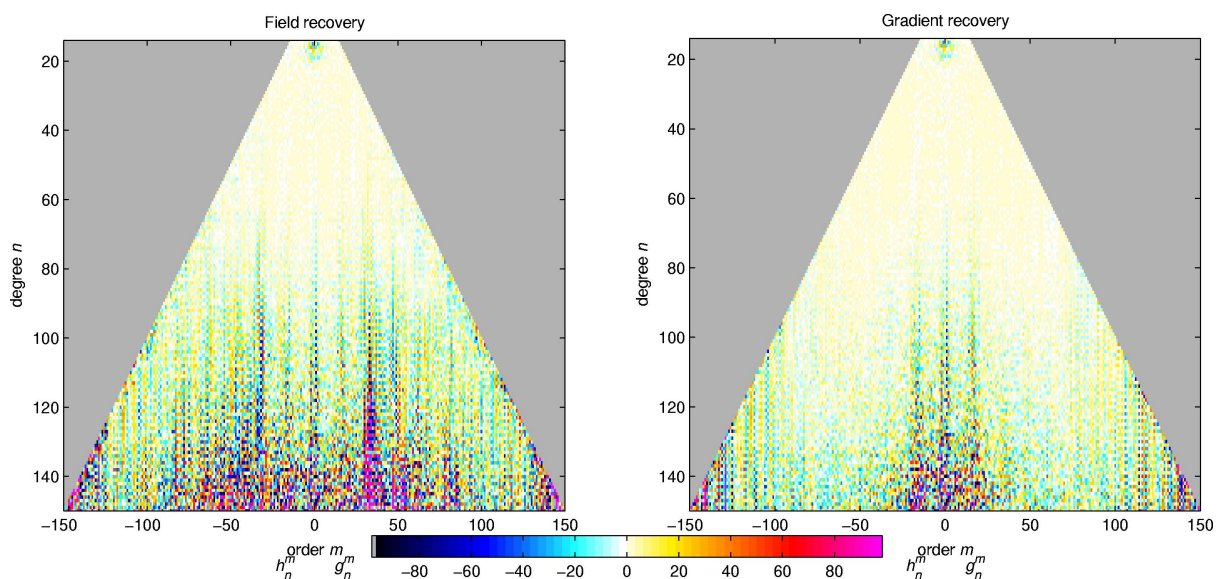


Figure 4.3: Sensitivity matrices for the lithospheric field recovery from CI applied directly to field values (left) and to low satellite sums and differences and high satellite data (right).

It is clear from the previous discussion that SIVW has successfully exploited the difference data from the low satellites to extract high order lithospheric field coefficients without degrading those of low order. Furthermore, Figs. 4.4 and 4.5 show that neither the magnetospheric nor the associated high-frequency induced coefficient recoveries have suffered. Not only do the magnetospheric coefficient time series show good agreement with the true series (blue), but the results from the traditional (red) and gradient approach (green) are identical. Notice, however, that certain recovered coefficients, such as q_1^1 , show large excursions from the true model at the beginning of the mission. This has been discussed by Sabaka and Olsen [2006] and is due to the minimal separation in local time of the low and high satellite orbital planes in co-rotation phase, which affects some coefficients. The high-frequency induced coefficients also show exact agreement between the two recovered time series. Again, there are large discrepancies between

certain recovered and true coefficients at the beginning and end of the mission due to minimal orbital plane separation. Indeed, it appears that SIVW applied to sums, differences, and high data to target high-order lithospheric recovery is performing precisely as hoped. That is, it is exploiting the enhanced signal to noise ratio of the high-order lithospheric field in the low satellite difference data to obtain superior recovery of this portion of the field without degrading the recovery of low-order lithospheric and non-lithospheric field parameters.

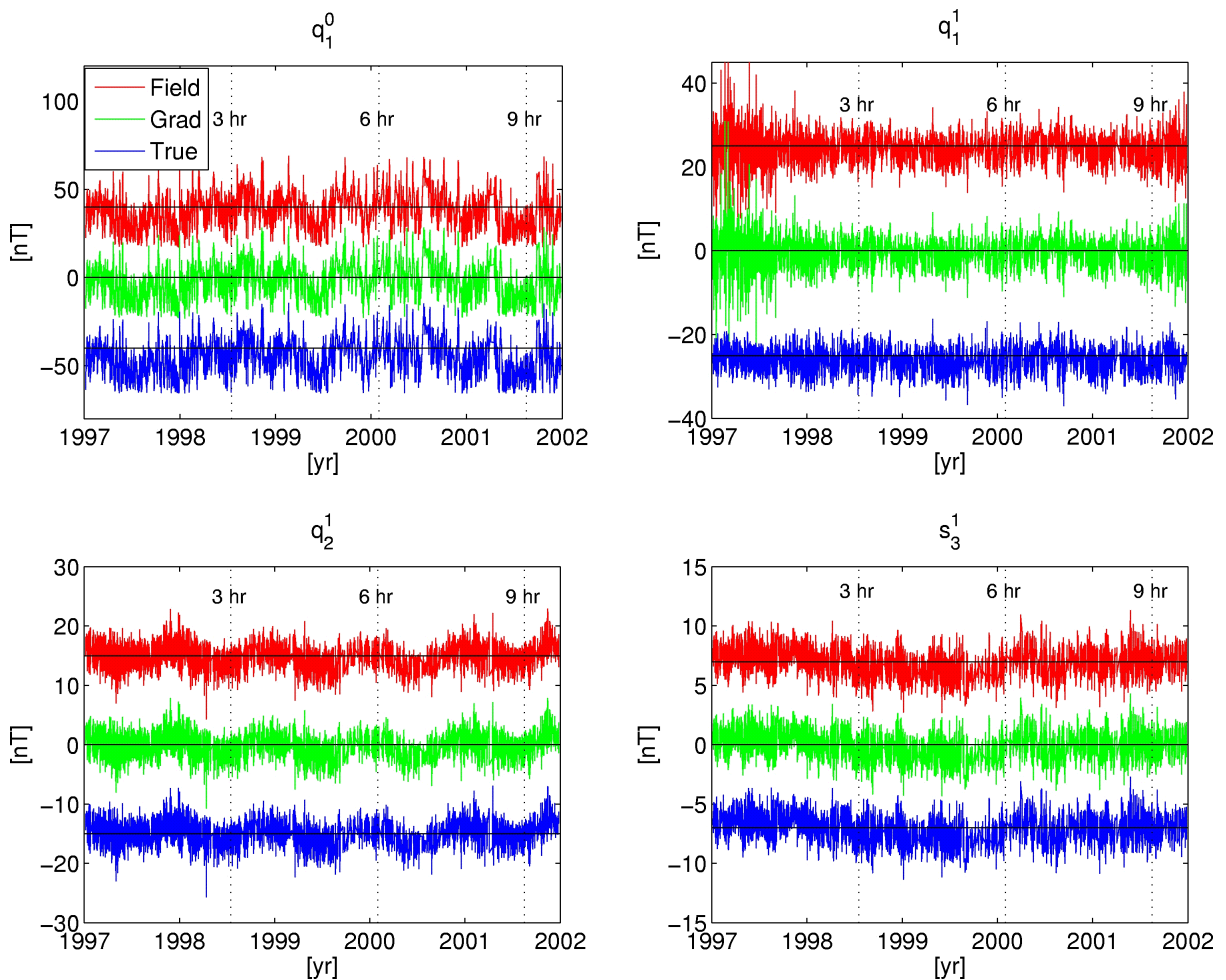


Figure 4.4: Time series of selected magnetospheric coefficients from the true model (blue) and from CI applied directly to field values (red) and to low satellite sums and differences and high satellite data (green). The blue and red lines have been shifted for clarity. Local time differences between the low and high satellite orbital planes are delineated by vertical dotted lines at 3, 6 and 9 hr separation.

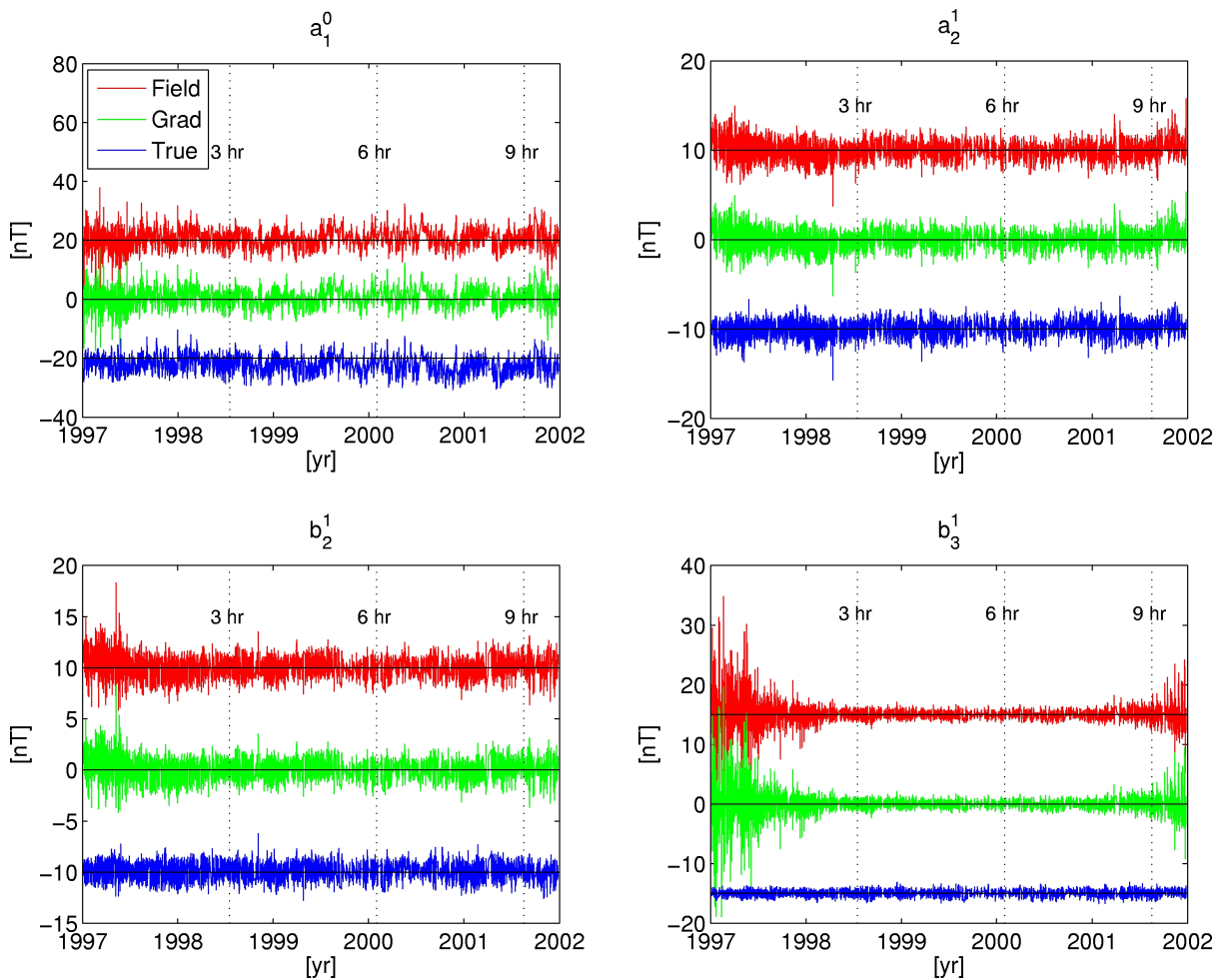


Figure 4.5: Time series of selected high-frequency induced coefficients from the true model (blue) and from CI applied directly to field values (red) and to low satellite sums and differences and high satellite data (green). The blue and red lines have been shifted for clarity. Local time differences between the low and high satellite orbital planes are delineated by vertical dotted lines at 3, 6 and 9 hr separation.

Chapter 5

Multi-Satellite In-Flight VFM/STR Alignment

This section describes the in-flight alignment of the *Swarm* magnetometers, i.e. the determination of the transformation between the coordinate frame of the vector magnetometer (VFM) and the one of the star camera (STR). This transformation is described by means of three Euler angles (α, β, γ) . Involved in this task are the working packages WP-2210 “Multi-Satellite Alignment: Algorithm Development”, and WP-2120 “Multi-Satellite Alignment: CI Implementation”. The described activity is an extension of the experiments that have been performed during Phase A [cf. section 4.2 Olsen et al., 2004].

To obtain the magnetic field in the North-East-Center (NEC) coordinate system, the VFM magnetometer readings (given in the VFM frame) are combined with the attitude measured by the STR. The latter measures the rotation between the International Celestial Reference Frame (ICRF) and the Common Reference System (CRF) of the optical bench; attitude is delivered in the CRF (the merging of the attitude of the individual camera heads to the CRF will not be discussed here), and therefore rotation between the CRF and the ICRF is straightforward. Likewise, transformation from the ICRF to the International Terrestrial Reference Frame (ITRF), and further to the North-East-Center Frame (NEC) is straightforward, too. Hence the following steps are required in the processing of the level 1b data:

$$\mathbf{B}_{\text{VFM}} \longrightarrow \mathbf{B}_{\text{CRF}} \longrightarrow \mathbf{B}_{\text{ICRF}} \longrightarrow \mathbf{B}_{\text{ITRF}} \longrightarrow \mathbf{B}_{\text{NEC}}$$

Note that no knowledge of the spacecraft attitude is required in this scheme; only the attitude of the optical bench (CRF) is required.

5.1 From VFM-Frame to NEC-Frame – Description of the Involved Rotations

Since a simulation of the star camera was not included in the present study, STR attitude data (i.e., attitude of the CRF) were not available. In order to obtain the attitude of the optical bench, it was therefore assumed that the CRF and spacecraft (SC) reference frames coincide, where the latter can be derived from the unit vector of the along-track velocity, but rotation was allowed between the CRF (i.e. SC frame) and the orbit reference frame (ORF). Note that this approach is only used for this simulation since no real STR attitude data were available. For the real processing of level 1b data, neither SC nor ORF attitude are required.

In order to transform the magnetic field vector $\mathbf{B}_{\text{NEC}} = (B_r, B_\theta, B_\phi)$ from the NEC coordinate system to that of the vector magnetometer, various rotations (between orthogonal coordinate systems) are necessary. In particular:

Rotation of the magnetic field from the NEC system to the orbit reference frame (ORF)

This rotation is given by

$$\mathbf{B}_{\text{ORF}} = \underline{\underline{R}}_1 \mathbf{B}_{\text{NEC}} \quad (5.1)$$

with

$$\underline{\underline{R}}_1 = \begin{pmatrix} \hat{x}_r & \hat{x}_\theta & \hat{x}_\phi \\ \hat{y}_r & \hat{y}_\theta & \hat{y}_\phi \\ \hat{z}_r & \hat{z}_\theta & \hat{z}_\phi \end{pmatrix} \quad (5.2)$$

where $\hat{\mathbf{x}} = (\hat{x}_r, \hat{x}_\theta, \hat{x}_\phi)$ is the unit vector of the x -direction in the ORF system. Figure 5.1 defines the coordinate system used here.

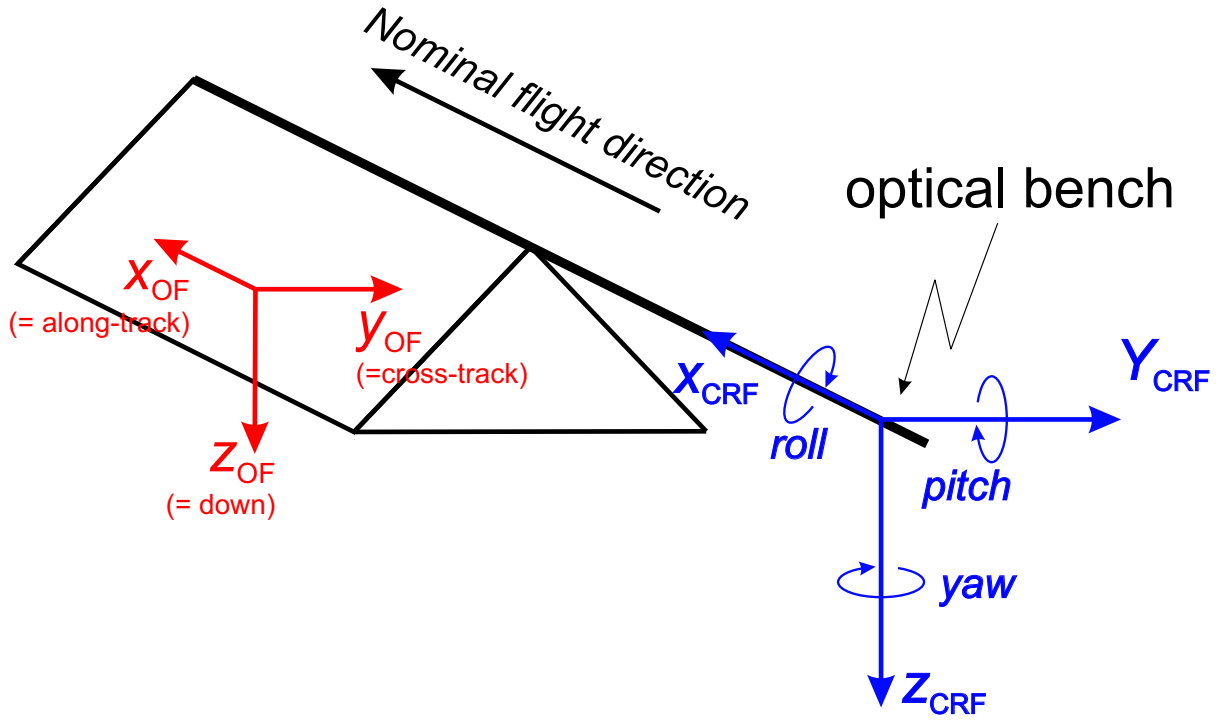


Figure 5.1: Definition of coordinate systems.

We assume that $\hat{\mathbf{x}}$ is in the (nominal) flight direction, so $\hat{\mathbf{x}} = \mathbf{v}/|\mathbf{v}|$ is determined from the instantaneous spacecraft velocity vector. Assuming that $\hat{\mathbf{z}}$ is in the plane spanned by $\hat{\mathbf{x}}$ and the radial direction (no roll misalignment of the spacecraft), we find

$$\hat{\mathbf{y}} = \frac{\hat{\mathbf{x}} \times \hat{\mathbf{r}}}{|\hat{\mathbf{x}} \times \hat{\mathbf{r}}|} = \frac{1}{\sqrt{x_\theta^2 + x_\phi^2}} \begin{pmatrix} 0 \\ x_\phi \\ -x_\theta \end{pmatrix}$$

$$\hat{\mathbf{z}} = \frac{\hat{\mathbf{x}} \times \hat{\mathbf{y}}}{|\hat{\mathbf{x}} \times \hat{\mathbf{y}}|} = \frac{1}{\sqrt{x_\theta^2 + x_\phi^2}} \begin{pmatrix} -(x_\theta^2 + x_\phi^2) \\ x_r x_\theta \\ x_r x_\phi \end{pmatrix}$$

Rotation of the magnetic field from the Orbital Frame (ORF) to the Common Reference Frame (CRF)

This rotation is given by

$$\mathbf{B}_{\text{CRF}} = \underline{\underline{R}}_2 \mathbf{B}_{\text{ORF}} \quad (5.3)$$

The Common Reference Frame (CRF) is the coordinate system of the optical bench that contains STR and VFM, and is defined by the merged attitude information of all three camera heads. Nominally the CRF is (approximately) equal to the orbit reference frame, and hence $\underline{\underline{R}}_2$ is a 3×3 identity matrix; however, since we are interested in the effect of rotating the optical bench with respect to the nominal flight direction, we modified the algorithm in order to parameterize $\underline{\underline{R}}_2$ by three Euler angles defining a 1-2-3 rotation.

Rotation of the magnetic field from CRF to the VFM system

This rotation is given by

$$\mathbf{B}_{\text{VFM}} = \underline{\underline{R}}_3 \mathbf{B}_{\text{CRF}} \quad (5.4)$$

where $\underline{\underline{R}}_3$ is parameterized by the three Euler angles (α, β, γ) as given by the product of three rotation matrices:

$$\underline{\underline{R}}_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix} \quad (5.5)$$

$$\underline{\underline{R}}_\beta = \begin{pmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{pmatrix} \quad (5.6)$$

$$\underline{\underline{R}}_\gamma = \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (5.7)$$

The algorithm has been modified to take into account different rotation schemes for each individual satellite of the *Swarm* constellation. The rotation matrix $\underline{\underline{R}}_3$ can be therefore expressed as the product of any of the three matrices (5.5) to (5.7), and a different combination is allowed for each satellite. We have performed our simulations assuming a 1-2-3 rotation, i.e.:

$$\underline{\underline{R}}_3 = \underline{\underline{R}}_\gamma \times \underline{\underline{R}}_\beta \times \underline{\underline{R}}_\alpha$$

The definition of this rotation is shown in Figure 5.2.

Estimation of the Euler angles

The Euler angles are estimated by comparing the observed magnetic vector components with those of a field model (that is co-estimated with the Euler angles). This approach is for instance described in Olsen et al. [2003]. (α, β, γ) are estimated in the CI-approach using a Least-Squares approach by minimizing

$$|\Delta \mathbf{B}_{\text{VFM}}| = \left| \mathbf{B}_{\text{VFM}}^{\text{obs}} - \mathbf{B}_{\text{VFM}}^{\text{mod}} \right|,$$

with

$$\begin{aligned} \mathbf{B}_{\text{VFM}}^{\text{obs}} &= \hat{\underline{\underline{R}}}_3 \underline{\underline{R}}_2 \underline{\underline{R}}_1 \mathbf{B}_{\text{NEC}}^{\text{obs}} \\ \mathbf{B}_{\text{VFM}}^{\text{mod}} &= \underline{\underline{R}}_3 \underline{\underline{R}}_2 \underline{\underline{R}}_1 \mathbf{B}_{\text{NEC}}^{\text{mod}} \end{aligned}$$

Here $\underline{\underline{R}}_3$ is the "true" CRF/VFM rotation (given by the true Euler angles to be computed) and $\hat{\underline{\underline{R}}}_3$ is its estimate. For the tests completed up to now, it has been assumed that the satellites are flying in nominal direction, and hence the rotation $\underline{\underline{R}}_2$ that brings from ORF to SC frame system is given by a 3×3 identity matrix. The magnetic field components computed in this simulation in the NEC coordinate system are therefore first rotated to the ORF (i.e. SC frame) system by $\underline{\underline{R}}_1$, and afterwards to the vector magnetometer frame by $\underline{\underline{R}}_3$.

The ability of the method to determine unbiased Euler angles depends on the approximation of the observed magnetic field $\mathbf{B}_{\text{VFM}}^{\text{obs}}$ by the model field $\mathbf{B}_{\text{VFM}}^{\text{mod}}$. However, not only the field difference, $\Delta \mathbf{B}_{\text{VFM}} = \mathbf{B}_{\text{VFM}}^{\text{obs}} - \mathbf{B}_{\text{VFM}}^{\text{mod}}$, but also the *properties* (nature) of this difference has impact on the Euler angles. Even a relatively large difference field $\Delta \mathbf{B}_{\text{VFM}}$ will only have

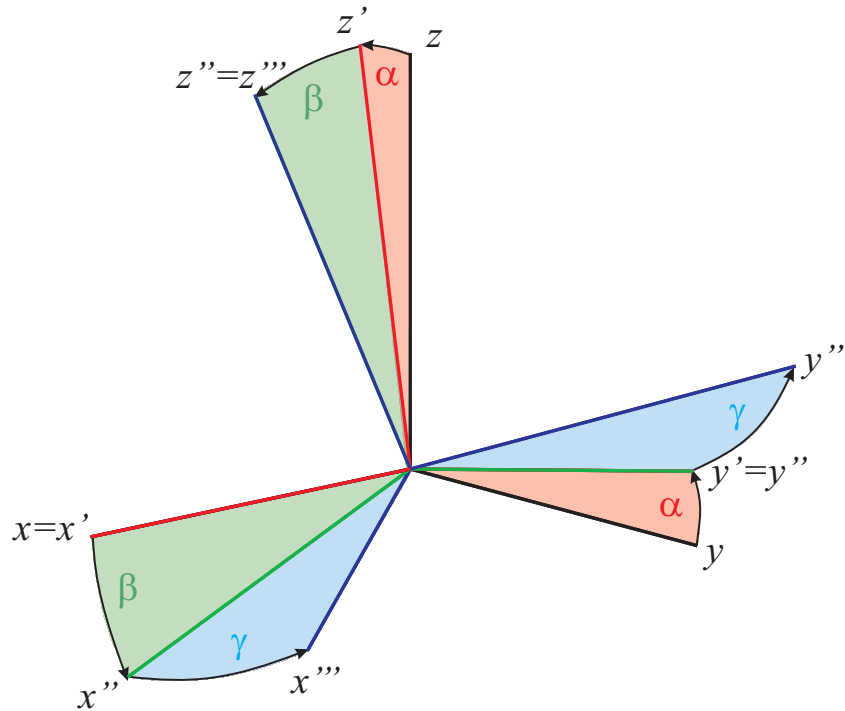


Figure 5.2: Definition of the three Euler angles α, β, γ describing a 1-2-3 rotation. Courtesy T. J. van Hord.

minor influence on the estimation of the Euler angles if $\Delta \mathbf{B}_{\text{VFM}}$ is a potential field, $\Delta \mathbf{B}_{\text{VFM}} = -\text{grad } \Delta V$. Also, a $\Delta \mathbf{B}_{\text{VFM}}$ with a *distribution* in the VFM frame that does not show a preferred direction will not harm the estimation of the Euler angles. The influence of certain *properties* of $\Delta \mathbf{B}_{\text{VFM}}$ on the estimation of the Euler angles was presented in the *Swarm* End-to-End Mission Performance Analysis Final Report [Olsen et al., 2004].

5.2 In-flight alignment using simulated data

The approach described in the previous section has been applied to data from the *Swarm* Constellation # 4. The data were selected according to the usual criteria to define geomagnetic quiet times for field modeling purposes: the global index of geomagnetic activity $K_p \leq 1^+$ for the time of observation and $K_p \leq 2^\circ$ for the previous three hour interval; and the absolute value for the index of magnetospheric ring-current strength, Dst , within 20 nT and its temporal variation less than 3 nT/hr. Data have not been selected neither for local time nor for geomagnetic latitude.

Data from all three satellites have been used for the tests, and all sets of Euler angles have been coestimated. A different set of Euler angles has been derived for each one of the three satellites of the constellation, closely following the approach presented for the present missions Ørsted and CHAMP [Olsen et al., 2003, 2006]. The next step in the analysis would be to do not only a simultaneous determination (i.e. coestimation) of the Euler angles of all *Swarm* satellites, but also to take more direct advantage of the constellation to perform this alignment. The true Euler angles that have been used to synthesize the data are time-independent and the same for all three satellites: $\alpha^{\text{true}} = +20$ arcsec, $\beta^{\text{true}} = +15$ arcsec, $\gamma^{\text{true}} = +10$ arcsec.

The different magnetic field contributions generated by the forward calculation scheme have

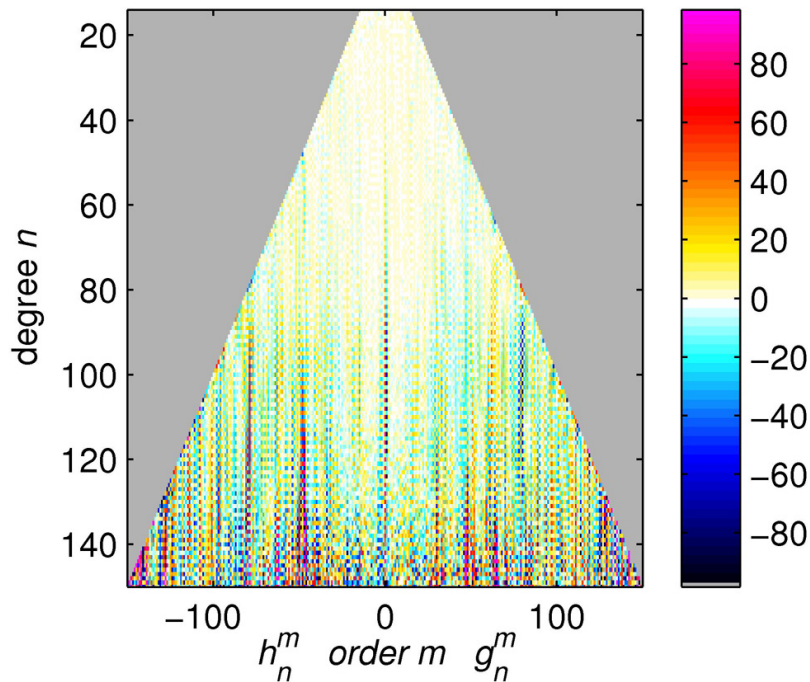


Figure 5.3: Sensitivity matrix for internal field coefficients. See text for details.

been added sequentially to generate diverse input data files in order to test the recovery of the Euler angles. Toroidal fields have not yet been included in the current tests, i.e. only Laplacian potential fields have been considered so far. The first tests involved synthetic data including only contributions from the core up to spherical harmonic degree $n = 13$, rotated from the NEC frame to the VFM frame using the true time-constant Euler angles. The model contained only internal coefficients up to the same degree (the time dependence of which is described by splines in the usual way) as well as the nine (three per satellite) Euler angles. The estimated internal coefficients were in excellent agreement with the true ones (differences smaller than a thousandth of nanoTesla) and the retrieved Euler angles were equivalent to the true Euler angles up to the last significant decimal place.

A second test used as input data the synthesized full contribution for the internal (core and crustal) field, i.e. up to degree $n = 150$. The purpose of this test was to check a possible contamination of the high degree/high order spherical harmonic coefficients due to the co-estimation of the Euler angles. The recovered coefficients are very close (differences of the order of tenths of nanoTesla) to the ones used for synthesizing the data. Regarding the Euler angles, the mean values for the (α, β, γ) Euler angles for all three satellites are less than 1 arcsec different from the true Euler angles.

Nex, we included the magnetospheric primary and induced field contributions, and solve for those. Figure 5.3 represents the sensitivity matrix for this case, and demonstrates that the inclusion of magnetospheric fields do not disturb the recovery neither of the Euler angles nor of the high degree crustal coefficients, at least in this simulation.

However, a different situation occurred when also ionospheric fields (and their induced counterpart) were included. From the preliminary results for that case we find that the Euler angles have not been recovered properly. There is in fact a convergence in the solution for the Euler angles but to wrong values. However, the recovery of the internal coefficients is only minimally

affected by the inclusion of the ionospheric fields, and the crustal field is recovered properly even in this case. We need to do some more work to understand why the inclusion of the ionospheric fields brings out a deficient recovery of the Euler angles, and also to check the effect of the inclusion of the toroidal contributions newly generated for Constellation # 4. We expect that a data selection that minimizes any preferred direction of $\Delta\mathbf{B}_{\text{VFM}}$, combined with the selective infinite variance weighting presented in chapter 4 will improve the results in order to minimize the effects of the inclusion of the ionospheric field into the Euler angle recovery.

We have also considered the case of time-varying Euler angles. Although this will be not the case for the *Swarm* mission (for which the Euler angles of the optical bench are required to be stable within 5 arcsec, according to the *System Requirements Document*, we investigated the robustness of the solution by introducing time-varying Euler angles. For this cases, data were split into bins of 1 month length during which the three Euler angles (α, β, γ) were considered constant, but they were allowed to vary sequentially by +1, -1, and +2 arcsec per bin, respectively. Data were therefore synthesized using these different Euler angle values and recovered afterwards by the algorithm.

A point to be emphasized is that the Euler angles were properly recovered without the need for data from the whole mission. Some tests have been developed using only input data for selected temporal intervals (the time span being chosen in order to have enough data for the recovery of the main and crustal components of the field up to a high degree expansion). Figure 5.4 shows the recovery of time-varying Euler angles for a particular test. In this case, only data for the last year of the mission were used. The input data and the model field contained internal contributions up to degree $n = 30$. The Euler angles were allowed to vary at one month interval, so a total of twelve sets of three Euler angles were obtained. As shown by the figure, the Euler angles were recovered with high accuracy. By averaging the differences between the twelve individual estimates and the *true* Euler angles we obtain $\Delta\alpha = 0.17$ arcsec, $\Delta\beta = 0.33$ arcsec, $\Delta\gamma = 0.31$ arcsec, with standard deviations σ of 0.13, 0.2, and 0.16 arcsec respectively, so the computed Euler angles are in very good agreement with the true ones.

Another important point is that there is apparently no influence on the value of the Euler angles used as starting point for the inverse algorithm. After few iterations the Euler angles are recovered with an accuracy of few tenths of arc second, no matter the initial values are set either equal to the right Euler angles (for example, those that would come for a pre-flight determination) or equal to zero. When ionospheric fields are included in the input data, however, the solution for the Euler angles converge to wrong values in the iterative solution scheme even when the initial values for the coefficients are set equal to the true ones.

Since the inclusion of ionospheric fields change so much the values of the Euler angles, which nevertheless converge in the iterative solution scheme, a question arises about the reliability of the in-flight determined Euler angles obtained for previous satellite missions.

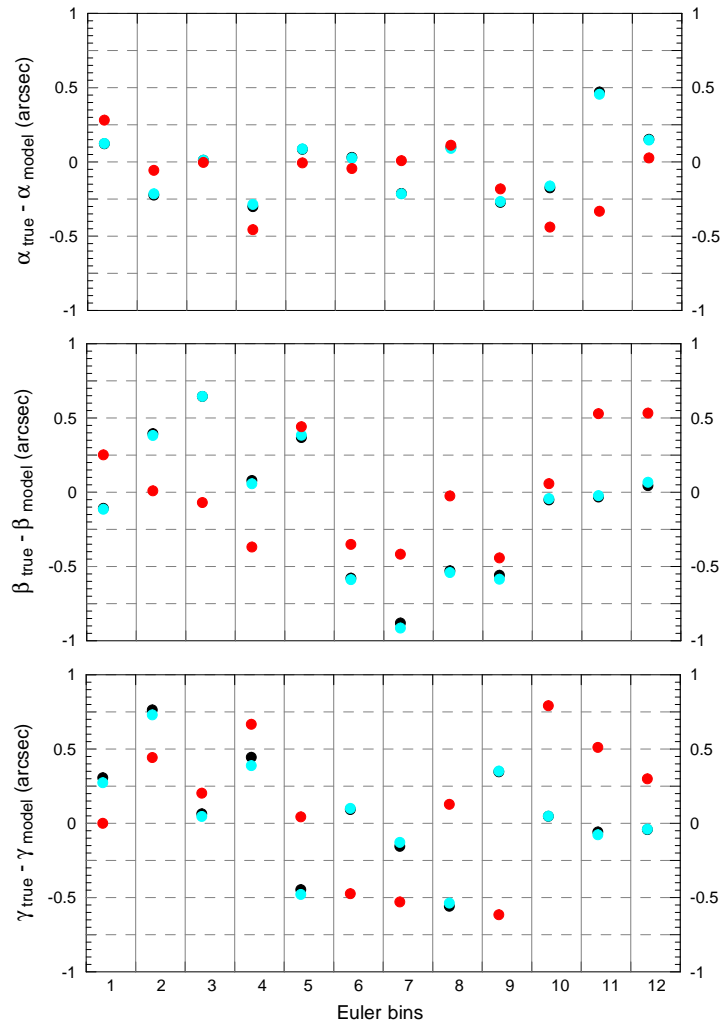


Figure 5.4: Difference between true and recovered time-varying Euler angles for *Swarm* 1 (black dots), 2 (blue dots), and 3 (red dots) satellites over one year. Each bin covers a time interval of one month.

Chapter 6

Failure and Imperfection Cases



Based on a note written by ESA in response to AI-004 of PM1, and a telecon on May 9, 2006, it was agreed to study the following failure and imperfection cases:

1. Failure of VFM and/or STR on a single satellite

- (a) Only scalar (no vector) data available for *Swarm A*
- (b) Only scalar (no vector) data available for *Swarm C*

2. Impact of a S/C magnetic field on a single satellite (*Swarm A*)

- (a) Constant S/C dipole moment (hard magnetization), corresponding to 2 nT at the location of the ASM
- (b) Induced S/C dipole moment (soft magnetization), corresponding to 3 nT at the location of the ASM over the poles (i.e. the area of maximum Earth's magnetic field strength)

Note: These two scenarii concern the magnetometer calibration, which is not part of the present study. Based on Ørsted and CHAMP experience, statistics of the magnetic field residuals (e.g. Table 1 of the CHAOS field model [Olsen et al. \[2006\]](http://www.dsri.dk/~nio/papers/CHAOS.pdf), available at <http://www.dsri.dk/~nio/papers/CHAOS.pdf>) can be used as a hint to see whether the magnetic data of one satellite are relatively biased compared to the other satellites. In other words: the present approach (single-satellite calibration) will in those cases tell us that “something is wrong with the data” (with no possibility to treat the data correctly and correct for the effect). Only a multi-satellite calibration approach (currently not foreseen as part of the present study) will be able to deal with these cases properly.

3. Noise in the CRF attitude of a single satellite (*Swarm A*)

- (a) Time dependent attitude noise (all components)

$$2 \sin(\omega_o t) \text{ arcsecs} + 10 \sin(2\pi T/24) \text{ arcsecs} \tag{6.1}$$

where t is UT, ω_t is orbital frequency, and T is Local Time in hours.

4. Failure of one or more satellite

- (a) Magnetic data from all 3 satellites (*Swarm A, B* and *C*)
- (b) Magnetic data from (*Swarm A* and *C*) only
- (c) Magnetic data from (*Swarm A* and *B*) only
- (d) Magnetic data from (*Swarm A*) only

Three of these scenarii have already been studied during Phase A; however, in order to obtain comparable results we decided to investigate all 4 cases using data of constellation # 4.

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