30552 Satellite Geodesy – E20

Lecture 2
Satellite orbits and Kepler elements

by
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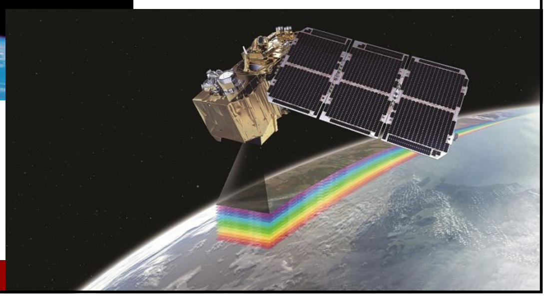
Motivation – why learn about satellite orbits?

- We need to:
  - know where the satellites are located
  - know how the satellites will move (future locations)
- Also for satellite geodesy:
  - Better knowledge of satellite location means more benefits of data collected by the satellites

Cryosat-2; ESA
GPS III; Lockheed Martin
Sentinel 2; Airbus



Outline

- Introduction to satellite orbits
  - The laws of Kepler
  - Orbit coordinate system
  - Conversion to CIS and CTS-systems
  - Perturbations

- Example: GPS satellite orbits
  - Kepler elements and orbit design
  - Orbit representation in broadcast ephemerids
  - Galileo and GLONASS orbits
  - Precise orbits

- GPS time
- Assignment 2

Fundamentals of celestial mechanics

- The two-body problem:
- "Given at any time the positions and velocities of two particles of known mass moving under their mutual gravitational force calculate their positions and velocities at any other time"

from G. Seeber, "Satellite Geodesy", 2003

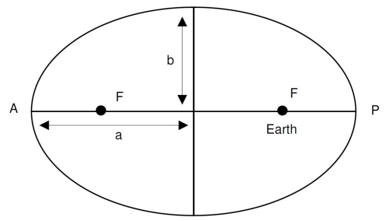
- Where the mass of one body, the satellite, is much smaller than the other body, the Earth

Kepler's Laws

Satellite orbits basically following the laws of Johannes Kepler (1571-1630), derived from observations of planets collected by Tycho Brahe (1546-1601)

1. **The orbit of each planet is an ellipse with the sun in one of the foci**

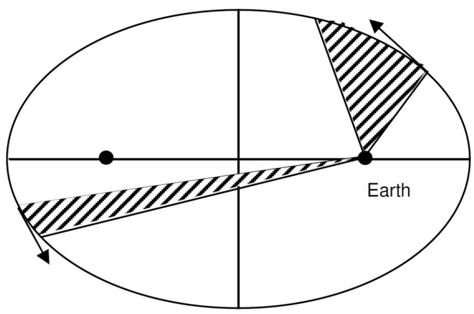
Consequence:
A satellite orbit is an ellipse with the gravitational center of the Earth in one of the foci



Kepler's Laws

2. The planets revolve with constant areal velocity, e.g. the radius vector of the planet sweeps out equal areas in equal lengths of time, independent of the location of the planet in the orbit

Consequence:
Satellites revolve with constant areal velocity



Kepler's Laws

3. **The relation between the square of the period, *T*, and the cube of the semi major axis, *a*, is constant for all planets:**

$$\frac{T^2}{a^3} = const$$

Consequence:

Two satellites with the same *a* will have the same *T* even if the eccentricities are different

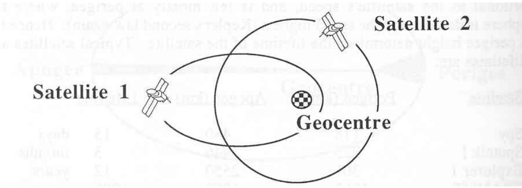


Illustration from:
"Guide to GPS Positioning", ed. by D. Wells, Canadian GPS Associates, 1987

Kepler's Laws

With the work of Isaac Newton (1642-1727) the constant from Kepler's 3. law can be determined:

$$\frac{T^2}{a^3} = const = \frac{4\pi^2}{GM} \quad => \quad T = \frac{2\pi}{\sqrt{GM}} a^{3/2}$$

T = orbital period; *a* = semi-major axis of the orbit;
M = mass of the Earth; *G* = universal gravitational constant
GM = 3986004.418*10^8 m^3/s^2

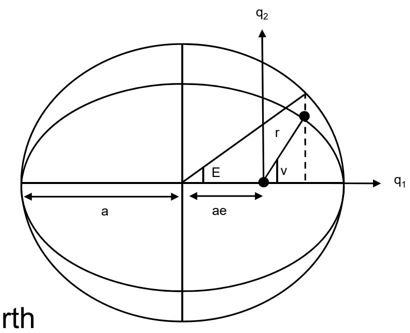
Kepler's laws would be true for all satellites if the Earth was a point mass, and if no other forces than Earth's gravity were affecting the satellites

Orbit coordinate system - definition

- A coordinate system for referencing of an SV position within the orbit

- Origo in center of the Earth
- 1.-axis, q_1, towards perigee
- 2.-axis, q_2, where v(t)=90º
- 3.-axis, q_3, is perpendicular to orbit plane (out of the figure)

- Perigee: Point of the orbit closest to the Earth

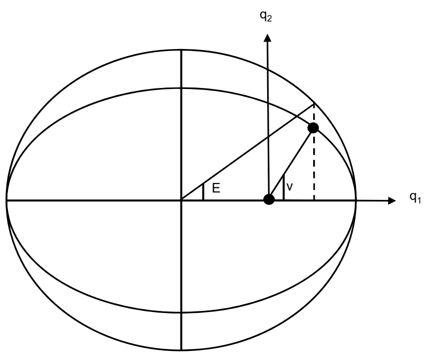


Angles to describe satellite motion (1)

- v(t), true anomaly, indicating satellite position on orbital ellipse

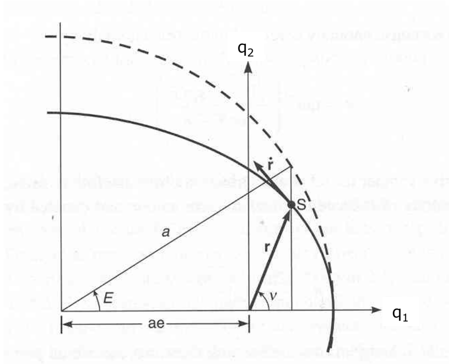
- E(t), eccentric anomaly, defined based on circumscribed circle

- M(t), mean anomaly, non-geometrical quantity based on mean motion, n of the satellite



Orbit coordinate system - terms

S(t), satellite position
a, semi major axis
e, eccentricity of ellipse
E(t), eccentric anomaly
v(t), true anomaly
r(t), position vector of S in orbit coordinate system
r'(t), velocity vector of S



Angles to describe satellite motion (2)

- When:
  t is current time, and t0 is time at perigee crossing
- Mean motion: $n = \sqrt{\mu}\, a^{-3/2}$ unit: radians/sec
- Mean anomaly: M(t) = n·(t – t0)
- Eccentric anomaly: E(t) = M(t) + e·sin(E(t))
- True anomaly: $v(t) = \arctan\left[\frac{\sqrt{1-e^2}\sin(E)}{\cos(E)-e}\right]$

Position vector in orbit coordinate system

- Satellite position in orbit coordinate system for a given time epoch is:

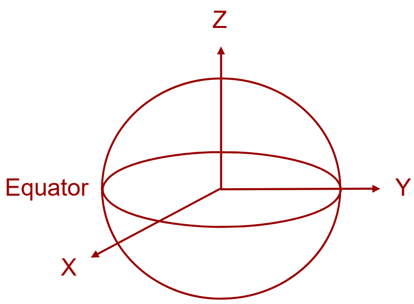
$$\mathbf{r} = \frac{a(1-e^2)}{1+e\cos v}\begin{bmatrix}\cos v\\ \sin v\\ 0\end{bmatrix} = \begin{bmatrix}a\cos E - ae\\ a\sqrt{1-e^2}\,\sin E\\ 0\end{bmatrix}$$

v and E are two different angles, both indicating the satellite position in the orbit as function of time

- The q_3 coordinate is zero when the satellite is in the orbit (ref. definition of the coordinate system)

Conventional Inertial Reference System (CIS)

- Origo at mass center of Earth
- X axis in Equatorial plane towards vernal equinox
- Z axis at mean rotational axis
- Y axis in Equatorial plane to form right handed cartesian system



- The system is fixed in space and does not rotate with the Earth

WGS84 – World Geodetic System 1984

- Origo at mass center of Earth
- Z-axis coincident with Rotational axis
- X-axis towards inter-section of Greenwich meridian with equatorial plane
- Y-axis to complete right hand coordinate system
- Position given as X, Y, Z or latitude, longitude, height

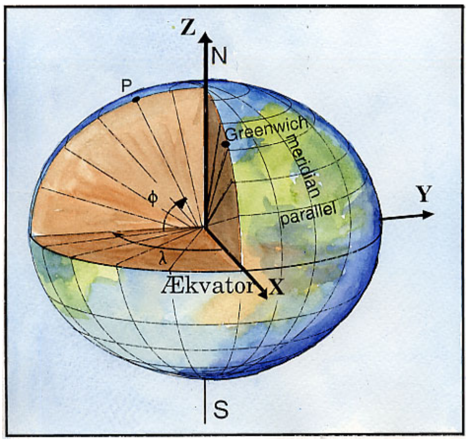


Figure from prof. Ole Jacobi

Satellite positions related to the Earth

- To determine the satellite position in a conventional terrestrial reference system (like WGS84) the inertial coordinate system (CIS) is used as an intermediate step
- The conversion is carried out by rotations of the orbit coordinate system with respect to the CIS
  - Note; the two systems have identical origo
- The rotation angles are:
  - Ω right ascension of the ascending node (RAAN)
  - i inclination
  - ω argument of perigee

Satellite orbits - Kepler elements

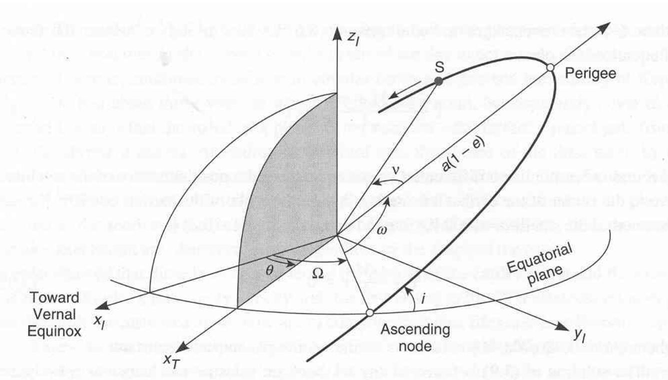


Figure from P. Misra and P. Enge, “Global Positioning System”, 2001

Conversion: orbit system -> CIS

The conversion is carried out by rotations of the orbit coordinate system, with index *q*, with respect to the CIRS, with index *x* (the two systems have identical origo):

$$\mathrm{r}_q = \mathrm{R}_{qx}\,\mathrm{r}_x \qquad \text{where} \qquad \mathrm{R}_{qx} = \mathrm{R}_3(\omega)\mathrm{R}_1(i)\mathrm{R}_3(\Omega)$$

and

$$\mathrm{R}_3(\Omega) = \begin{bmatrix} \cos(\Omega) & \sin(\Omega) & 0 \\ -\sin(\Omega) & \cos(\Omega) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{R}_1(i) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(i) & \sin(i) \\ 0 & -\sin(i) & \cos(i) \end{bmatrix}$$

$$\mathrm{r}_x = \mathrm{R}_{xq}\,\mathrm{r}_q \qquad \mathrm{R}_{xq} = \mathrm{R}_3(-\Omega)\mathrm{R}_1(-i)\mathrm{R}_3(-\omega)$$

Conversion: orbit system -> CIS -> CTS

- The first conversion is carried out by rotations of the orbit coordinate system, q, with respect to the CIS, X_I, (the two systems have identical origo):

$$\mathbf{r}_{X_I} = \mathbf{R}_{X_I q}\,\mathbf{r}_q \qquad \text{where} \qquad \mathbf{R}_{X_I q} = \mathbf{R}_3(-\Omega)\mathbf{R}_1(-i)\mathbf{R}_3(-\omega)$$

The second conversion from CIS, X_I to CTS, X_T is carried out adding an extra rotation around the Z-axis, with the Greenwich sideral time, Θ:

$$\mathbf{r}_{X_T} = \mathbf{R}_3(\Theta)\mathbf{r}_{X_I}$$

Satellite orbits - Kepler elements

- Orbit size and shape:
  - a semi major axis
  - e eccentricity
- Location of orbit relative to Earth:
  - i inclination
  - Ω right ascension of the ascending node
  - ω argument of perigee
- Location of satellite in the orbit:
  - v true anomaly or E - eccentric anomaly

Perturbations of satellite orbits

Kepler's laws would be true for all satellites if the Earth was a point mass, and if no other forces than Earth's gravity were affecting the satellites

So, Kepler's laws are not true in reality.

Discuss with in small groups (2-3 minutes):

- What may impact the motion of a satellite in its orbit?

There are several things, we list them on the board after your discussion

Size of perturbations of GPS satellite motion – if not accounted for

Perturbation source	Effect on GPS orbit after 2 hours	Effect on GPS orbit after 3 days
Deviation of Earth gravity field from spherical shape	2 km	14 km
Other variations in Earth gravity field	50 – 80 meter	100 – 500 meter
Solar and lunar gravitation	5 – 150 meter	1 – 3 km
Earth body tides	-	0.5 – 1.0 meter
Ocean tides	-	0.0 – 2.0 meter
Solar radiation pressure	5 – 10 meter	100 – 800 meter
Albedo	-	1.0 – 1.5 meter

Source: G. Seeber, "Satellite Geodesy", 2nd edition, de Gruyter

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Kepler elements for GPS

Table 3.4 Orbital parameters for the baseline 24-satellite GPS constellation
Semi-major axis: 26,559.8 km; Eccentricity: 0; Inclination: 55°; Argument of Perigee: 0
[from Massatt and Zeitzew (1998)]

Slot ID	*Right Ascension (deg)*	*Mean Anomaly (deg)*	*Slot ID*	*Right Ascension (deg)*	*Mean Anomaly (deg)*
A3	272.85	11.68	D1	92.85	135.27
A4	272.85	41.81	D4	92.85	167.36
A2	272.85	161.79	D2	92.85	265.45
A1	272.85	268.13	D3	92.85	35.16
B1	332.85	80.96	E1	152.85	197.05
B2	332.85	173.34	E2	152.85	302.60
B4	332.85	204.38	E4	152.85	333.69
B3	332.85	309.98	E3	152.85	66.07
C1	32.85	111.88	F1	212.85	238.89
C4	32.85	241.57	F2	212.85	345.23
C3	32.85	339.67	F3	212.85	105.21
C2	32.85	11.80	F4	212.85	135.35

Figure: P. Misra and P. Enge, "Global Positioning System", 2001

Kepler elements for GPS

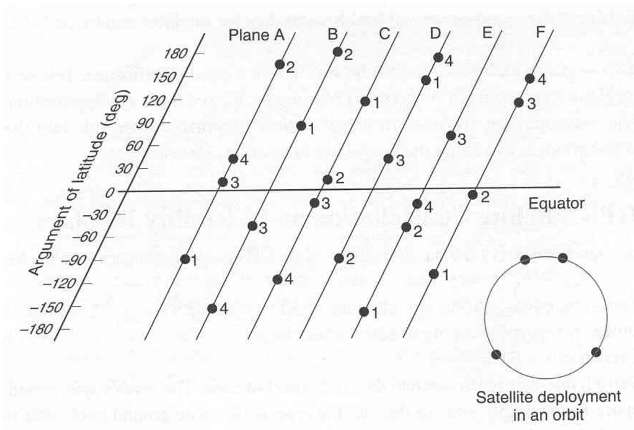
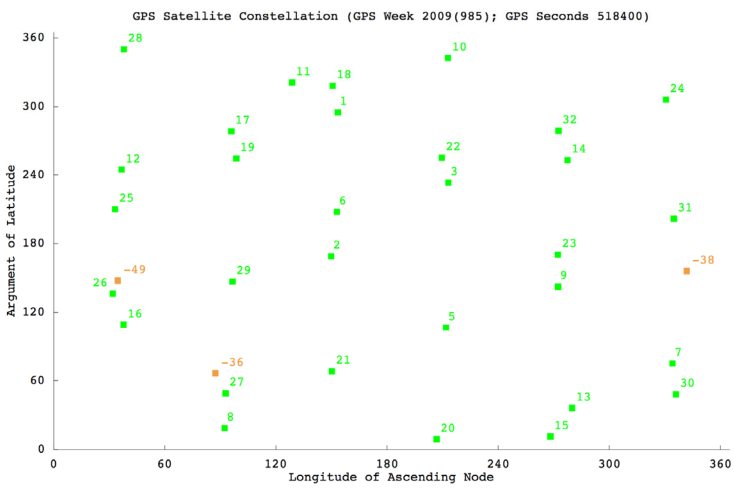


Figure: P. Misra and P. Enge, "Global Positioning System", 2001

Distribution of GPS satellites in orbit planes – in reality



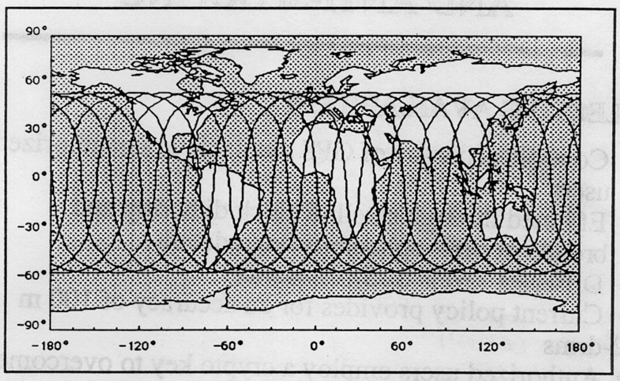
Plot from Prof. Richard Langley, University of New Brunswick, Canada

GPS satellite outages

- Information about the current satellite constellation is available from the US Coast Guard web site: https://www.navcen.uscg.gov/?pageName=gpsAlmanacs
  – These are ascii text files
  – Visualisations of the files can be found on the internet e.g. on this site:
    http://navigationservices.agi.com/SatelliteOutageCalendar/SOFCalendar.aspx

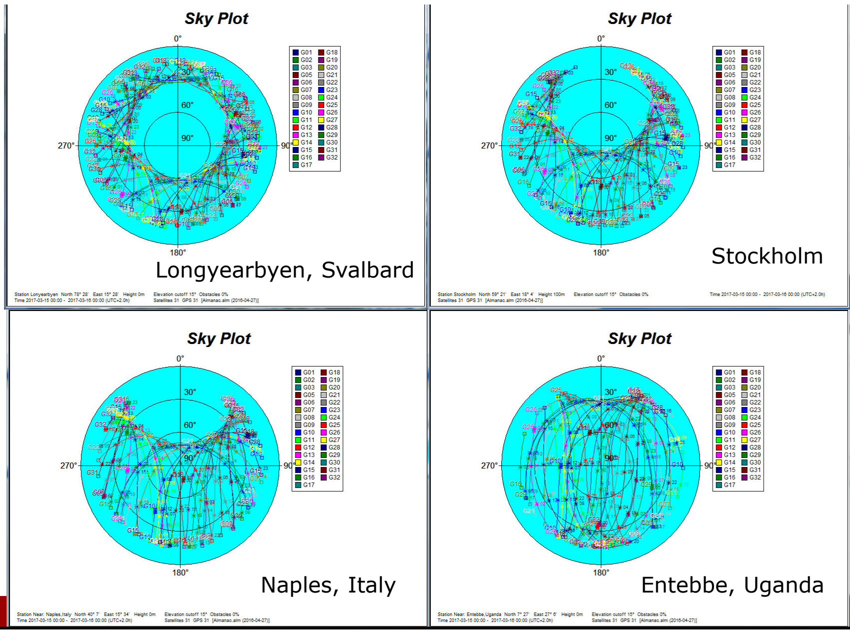
- Similar information on operational status can be found for most satellites and satellite missions where the data is available to the public

GPS satellite ground track



Typical Ground Tracks (After Santerre, 1989)

GPS satellite constellation - 24 hours



GPS Broadcast ephemeris

- Contains:
  - Reference time, t0
  - Kepler elements given as:
    - √a, e, i at t0
    - Ω at beginning of the GPS week
    - ω and M at t0
  - Correction factor to the mean motion, Δn
  - Rate of change of i and Ω
  - Coefficients for correction models to ω, a and i
- The GPS receiver uses this information to determine satellite positions as X, Y, Z coordinates in WGS84

GPS Navigation Message

- Content:
  - Hand over Word – current week second
  - Almanac (approximate positions for all satellites)
  - Coefficients for ionospheric model
  - Offset between GPS-time and UTC-time
- And for each satellite:
  - Broadcast ephemeris
  - Clock corrections and satellite health

Use of broadcast ephemeris in GPS receiver

- The information given with the broadcast ephemeris are used in the GPS receiver for the following steps:
  - Estimate satellite position in orbital coordinate system at time of signal transmission
  - Convert position to inertial system
  - Convert position to WGS84 coordinate system
  - Generate observation equation for each satellite and solve for receiver position

$$R_r^s = \sqrt{(X_r - X^s)^2 + (Y_r - Y^s)^2 + (Z_r - Z^s)^2} + c \cdot \Delta\delta_r$$

Algorithm for computation of satellite coordinates from broadcast navigation message

This is provided in the GPS Interface Control Document (the ICD), an official document issued by the US DoD

$\mu = 3.986005 \times 10^{14}$ meters^3/sec^2	WGS 84 value of the earth's universal gravitational parameter
$\dot{\Omega}_e = 7.292115167 \times 10^{5}$ rad/sec	WGS 84 value of earth's rotation rate
$a = \left(\sqrt{a}\right)^2$	Semimajor axis
$n_0 = \sqrt{\frac{\mu}{a^3}}$	Computed mean motion - rad/sec
$t_k = t - t_{oe}^{*}$	Time from ephemeris reference epoch
$n = n_0 + \Delta_n$	Corrected mean motion
$M_k = M_0 + n t_k$	Mean anomaly
$M_k = E_k - e \sin E_k$	Kepler's equation for eccentric anomaly
$\cos f_k = (\cos E_k - e)/(1 - e \cos E_k)$, $\sin f_k = \sqrt{1-e^2}\,\sin E_k / (1 - e\cos E_k)$	True anomaly
$E_k = \cos^{-1}\left[\frac{e + \cos f_k}{1 + e \cos f_k}\right]$	Eccentricity anomaly
$\phi_k = f_k + \omega$	Argument of latitude
$\delta u_k = C_{us} \sin 2\phi_k + C_{uc} \cos 2\phi_k$	Argument of latitude correction (2nd harmonic pertubations)
$\delta r_k = C_{rc} \cos 2\phi_k + C_{rs} \sin 2\phi_k$	Radius corrections (2nd harmonic pertubations)
$\delta i_k = C_{ic} \cos 2\phi_k + C_{is} \sin 2\phi_k$	Correction to inclination (2nd harmonic pertubations)
$u_k = \phi_k + \delta u_k$	Corrected argument of latitude
$r_k = a(1 - e \cos E_k) + \delta r_k$	Corrected radius
$i_k = i_0 + \delta i_k + (\mathrm{IDOT}) t_k$	Corrected inclination
$x_k' = r_k \cos u_k$, $y_k' = r_k \sin u_k$	Positions in orbital plane
$\Omega_k = \Omega_0 + (\dot{\Omega} - \dot{\Omega}_e) t_k - \dot{\Omega}_e t_{oe}$	Corrected longitude of ascending node
$x_k = x_k' \cos \Omega_k - y_k' \cos i_k \sin \Omega_k$, $y_k = x_k' \sin \Omega_k + y_k' \cos i_k \cos \Omega_k$, $z_k = y_k' \sin i_k$	Earth fixed coordinates

* t is GPS system time at time of transmission, i.e., GPS time corrected for transit time (range/speed of light). Furthermore, t_k shall be the actual total time difference between the time t and the epoch time t_{oe}, and must account for beginning or end of week crossovers. That is, if t_k is greater than 302,400, subtract 604,800 from t_k. If t_k is less than -302,400 sec, add 604,800 sec to t_k.

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- ✓Orbit coordinate system
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- ✓Perturbations

✓Example: GPS satellite orbits
- ✓Kepler elements and orbit design
- ✓Orbit representation in broadcast ephemerids
- – Galileo and GLONASS orbits
- – Precise orbits

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Galileo satellite orbits – Kepler elements

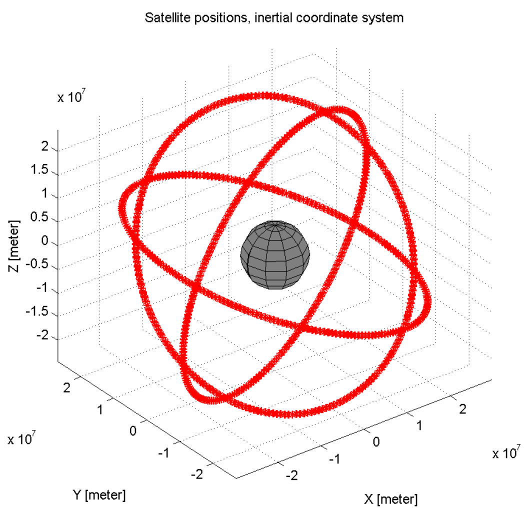
- Orbit size and shape:
  - a semi major axis: 29600.318 km
  - e eccentricity: 0.0

- Location of orbit relative to Earth:
  - i inclination: 56°
  - Ω right ascension of the ascending node: 0°, 120°, 240°
  - ω argument of perigee: 0.0°

- Location of satellite in the orbit:
  - M mean anomaly (27 sv):
    - 0.00° - 320.00° in 40° steps
    - 13.33° - 333.33° in 40° steps
    - 26.66° - 346.66° in 40° steps

Source: Galileo Project Office, ESA

Galileo orbits - simulation



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GLONASS satellite orbits – Kepler elements

- Orbit size and shape:
  - a semi major axis: 25440 km
  - e eccentricity: 0.0
- Location of orbit relative to Earth:
  - i inclination: 64° 8’
  - Ω right ascension of the ascending node: 0°, 120°, 240°
  - ω argument of perigee: 0.0°
- Location of satellite in the orbit:
  - M mean anomaly:
    - Even spacing in orbits, i.e. 45° spacing with 24 satellites

Source: B. Forssell:
Technical Comparison between the GLONASS and GPS Concepts. In *Geodetic Applications of GPS*. 1997

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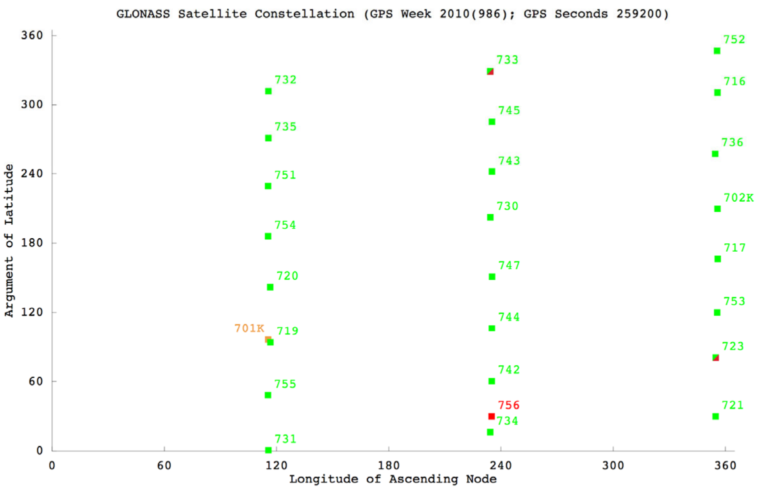
GLONASS orbits – real satellite positions



Plot by Prof. Richard Langley, University of New Brunswick, Canada

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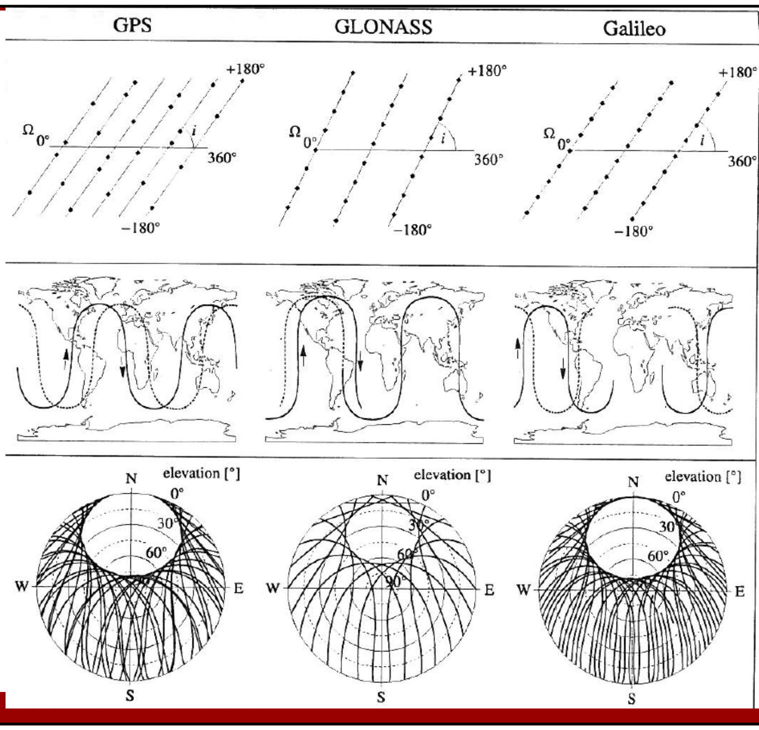

From: "GNSS…"
by Hoffmann-Wellenhof,
Lichtnegger and Wasle,
Springer, 2008

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Orbit determination and precise orbits

- For systems like GPS, GLONASS and Galileo predicted satellite orbits are used for real time positioning
- Satellite positions and orbits can also be calculated in post mission, i.e. after data from the satellite has been collected.
- This is referred to as orbit determination. Such satellite positions are more precise, thus precise orbits
- In the lecture next week, orbit determination and precise orbits will be discussed

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- GPS time
- Assignment 2

GPS time (1)

- For GPS-positioning it is crucial with a homogeneous *and* continuous time scale. Therefore a new time scale was defined to be used for GPS, called *GPS time*

- GPS time is defined as UTC but without the leap seconds
  - GPS time = UTC time at January 6th, 1980, @ 00:00 hours
  - GPS time in 2020 equals UTC + 18 seconds + a few nanoseconds

- GPS time is maintained by the US Naval Observatory http://www.navcen.uscg.gov

GPS time (2)

- A time epoch given in GPS time is referenced as:
  - Week number, counted since January 6th, 1980
  - Number of seconds within the week (counter is reset at midnight between Saturday and Sunday, i.e. from 604800 to 0 seconds)

- Current GPS week is: 2120

- For some applications, GPS time is also given as:
  - Week number, day of week, seconds of day

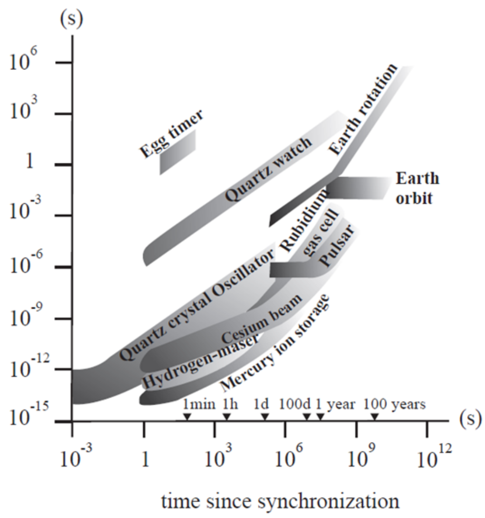
Stability of clocks and frequency standards

- A precise and well defined time reference, only makes sense if we have clocks sufficiently good to "keep the time"

- For example for GPS:
  An error of 1 microsecond in the GPS receiver clock induces an error of 300 meter in the range estimate to the satellite

- Estimation and modelling of both satellite and receiver clock errors is therefore important for geodetic applications

- Figure: G. Seeber, "Satellite Geodesy", 2003



GPS satellite clock error (or instability)

- GPS satellites are equipped with two (or three) rubidium and two cesium atomic clocks that are monitored by the GPS control stations
- The current size of the satellite clock errors are between 2 and 750 microseconds
- Satellite clock errors are modelled by polynomials of second order. Coefficients are transmitted to GPS users via the navigation message from the satellites

GPS satellite clock error model (1)

- Satellite clock correction model valid at time t:

$$\delta t = a_{f0} + a_{f1}(t - t_0) + a_{f2}(t - t_0)^2 + \Delta t_r$$

- Where:
  - t_0 is the reference time epoch
  - a_{f0} is coefficient for clock offset (seconds)
  - a_{f1} is coefficient for fractional offset (sec/sec)
  - a_{f2} is coefficient for clock drift (sec/sec^2)
  - Δt_r is related to the relativistic effect

GPS satellite clock error model (2)

- There is also a relativistic effect of the satellite clocks since the satellites move with respect to the earth. Compensation is handled by running the satellite clocks a little slower

- When the model of satellite clock correction is applied to all satellites, their clocks are synchronized to within 5-10 nanoseconds

Practical considerations

- For many applications it is important to remember the difference between especially UTC and GPS time. For instance:
  - When logging GPS data to a PC which runs on UTC time
  - When integrating GPS with other sensors such as cameras, laser scanners or INS equipment that might be referenced to UTC

- Example:
  - An airplane, with a speed of 500 km/h, will move 2,500 meters in 18 seconds
  - When not correcting for the time offset between UTC and GPS time, an equivalent error in the position is introduced

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Assignment 2

- Input needed:
  - Nominal Kepler elements for GPS and Galileo satellites
  - Equations which are given in the text book and slides from Lecture 1 and Lecture 2
  - Matlab scripts from Assignment 1

- Do:
  - Determine q_1, q_2, q_3 coordinates of GPS satellites in orbital system
  - Determine X, Y, Z coordinates of the satellites in CIS
  - Plot the X, Y, Z coordinates, and visually evaluate the result
  - Implement a time counter; update the mean anomaly via the mean motion, estimate satellite positions in orbital system, and convert to inertial system
  - Plot the X, Y, Z coordinates of the satellites during 12 hours, and evaluate if the orbits are circular

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Assignment 2

- Then do:
  - The same for the Galileo satellites, nominal constellation
- Finally:
  - Use both GPS and Galileo satellites
  - Convert satellite positions to CTS
  - Define local ellipsoidal coordinate system and convert satellite positions to this system
  - Determine number of visible satellites (above local horizon) for given point in time

- Tip:
- The mean anomaly is given with the Kepler elements. It must be converted to eccentric or true anomaly