Inversion for quasi-geostrophic modes of core flow

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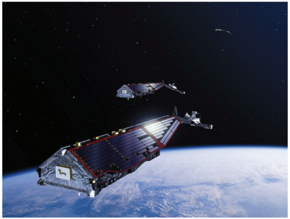
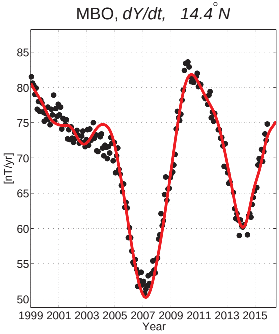
With thanks to Nicolas Gillet and Olivier Barrois

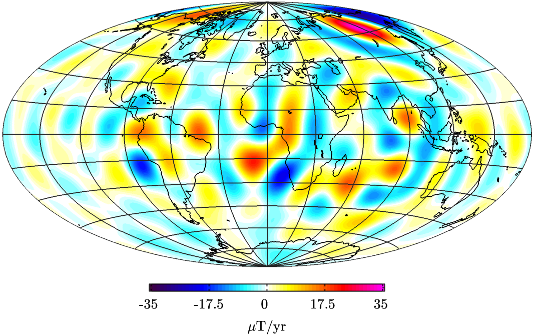
L'équipe Géodynamo Grenoble: 20 ans!

N. Gillet, circa 2006; L'équipe Géodynamo ski excursion, 2010.

Motivation: Secular variation at low latitudes

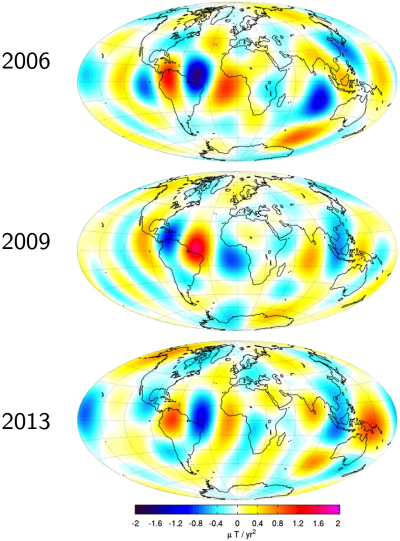


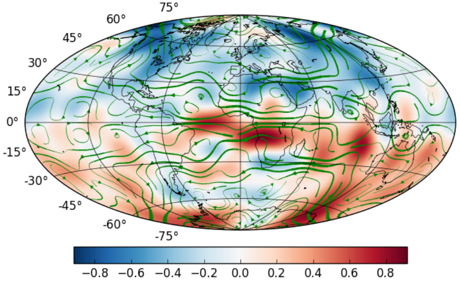


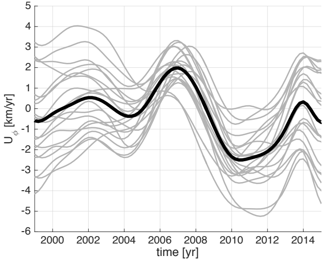


- Modern geomagnetic data: ground observatories & satellite data (CHAMP, now *Swarm*)
- Strong secular variation at latitudes below 30 degrees, esp. under Atlantic hemisphere

Motivation: Secular variation at low latitudes







- Localised secular acceleration pulses *[Lesur et al., 2008, Olsen and Mandea, 2008]*
- Compatible with non-axisymmetric azimuthal flow fluctuations *[Gillet et al., 2015]*

Motions in a rapidly-rotating sphere

- Consider an inviscid, incompressible, fluid in rapid rotation and subject to an impenetrable spherical boundary condition *[Greenspan (1969)]*

$$\frac{\partial \mathbf{u}}{\partial t} + 2\mathbf{e}_z \times \mathbf{u} = -\nabla p, \qquad \nabla \cdot \mathbf{u} = 0, \qquad \mathbf{e}_r \cdot \mathbf{u} = 0 \text{ at r=1.} \tag{1}$$

- 'Inertial' mode solutions to this problem take the form *[Zhang et al., (2001)]*

$$\mathbf{u}_I = \hat{\mathbf{u}}_I^{M,N,n}(s,z)\,\mathrm{e}^{i(M\phi+2\mathrm{i}\sigma t)}, \tag{2}$$

for both E^S and E^A symmetries, characterised by 3 integers M, N and n.

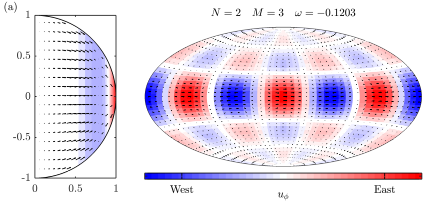
- The slowest E^S inertial modes, for given M, N are known as 'quasi-geostrophic' modes.
- To form a complete basis, the inertial modes should be supplemented by geostrophic modes, for example of the form *[Liao and Zhang (2010); Ivers et al., (2015)]*

$$\mathbf{u_G} = G_{2K-1}(s)\,\mathbf{e}_\phi \quad \text{where} \quad G_{2K-1} = \sum_{j=1}^{K} \frac{(-1)^{K-j}[2(K+j)-1]!!}{2^{K-1}(K-j)!(j-1)!(2j)!!} s^{2j-1} \tag{3}$$

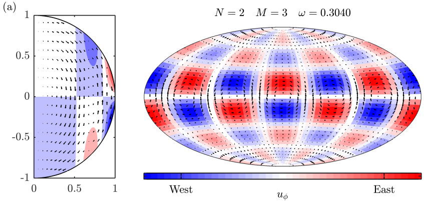
Example modes in a rapidly-rotating sphere

DTU

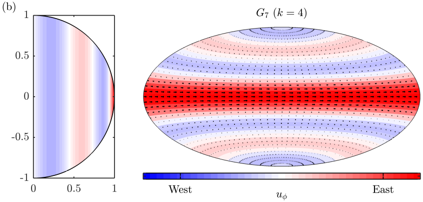
Symmetric inertial mode



Anti-symmetric inertial mode



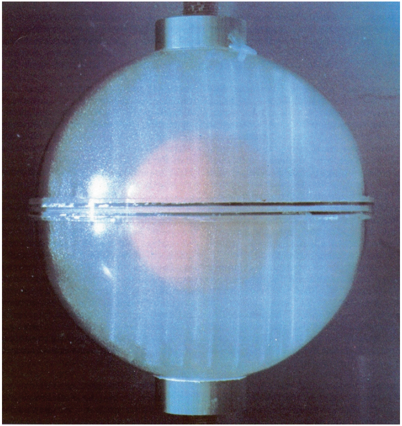
Geostrophic mode

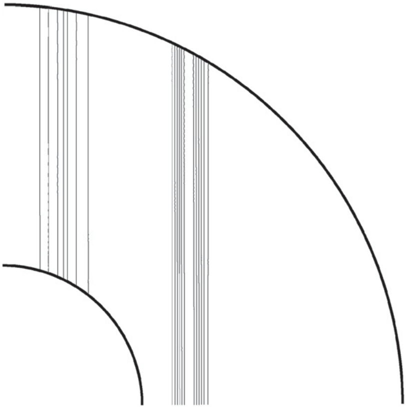


- Modes are in general 3D (although slowest E^S modes are quasi-2D)
- Fully compatible with spherical geometry
- Well suited for study of equatorial dynamics

Quasi-Geostrophic (QG) modes may dominate?

- In rapidly-rotating homogenous fluids, response to slow perturbations is often columnar *[Hide (1966); Busse & Carrigan (1976); Zhang and Liao (2004); Jault (2008); Bardsley & Davidson, (2016)]*





- Projection onto QG modes is a promising way towards reduced model of core dynamics *[Labbé et al., 2015]*

Idea here: Try to invert SV for a flow consisting of QG modes

Forward problem

- Writing $B_r = \overline{B}_r + \widetilde{B}_r$, sum of known large-scale and the unknown small-scale field, the diffusionless induction equation for the large-scale SV at the core surface is

$$\frac{\partial \overline{B}_r}{\partial t} = -\overline{\nabla_{\mathrm{H}} \cdot \left(\mathbf{u}\overline{B}_r\right)} + e, \qquad \text{where} \qquad e = -\overline{\nabla_{\mathrm{H}} \cdot \left(\mathbf{u}\widetilde{B}_r\right)} \tag{4}$$

- **Expand the instantaneous flow $\mathbf{u}$ as a sum of geostophic and QG inertial modes**

$$\mathbf{u}(s,z,\phi) = \sum_{K=1}^{K_{max}} a_G^K \, G_{2K-1}(s)\, \mathbf{e}_\phi \quad + \quad \sum_{M=1}^{M_{max}} \sum_{N=1}^{N_{max}} a_I^{M,N} \, \hat{\mathbf{u}}_I^{M,N}(s,z) \mathrm{e}^{iM\phi} \tag{5}$$

- Here, we consider only E^S modes with lowest frequency (i.e. the QG modes)
- Chose truncation levels $K_{max} = 20$, $M_{max} = 16$, $N_{max} = 10$

- Large-scale field $\overline{B}_r$ is up to SH deg. 14, e also parameterized in SH up to deg. 14
- When evaluating (4) only need the core surface part of the flow

- Solve (4) via vector spherical transform using $\mathbf{u}$ in poloidal-toriodal form, up to deg. 39

Inverse problem

- Assume $\overline{B}_r$ at CMB is known 1999-2016 (taken from CHAOS-6 field model)
- Data $\mathbf{d}$: $\frac{\partial \overline{B}_r}{\partial t}$ on grid outside tangent cylinder at CMB; assumed error 2μT/yr.
- Solve for mode amplitudes $\mathbf{a} = \left\{a_G^K; a_I^{M,N}\right\}$ defining large-scale flow
- And simultaneously for SH coefficients of SV due to unresolved scales $\mathbf{e} = \{e_t^s\}$

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- Regularized least-square inversion for $\mathbf{m} = \{\mathbf{a}; \mathbf{e}\}$. Seek to minimize

$$\Phi = (\mathbf{d} - \mathbf{Hm})^T \mathbf{C_d}^{-1} (\mathbf{d} - \mathbf{Hm}) + \lambda \mathbf{R}(\mathbf{a}) + \mathbf{e}^T \mathbf{P}_{e,e}^{-1} \mathbf{e} \tag{6}$$

$\mathbf{H}$ is the matrix connecting $\mathbf{d}$ to $\mathbf{m}$
$\mathbf{R}(\mathbf{a})$ is a norm measuring some property of the large-scale flow
$\mathbf{P_{e,e}}$ is prior covariance matrix of $\mathbf{e}$ from a geodynamo simulation *[Barrois et al., 2016]*

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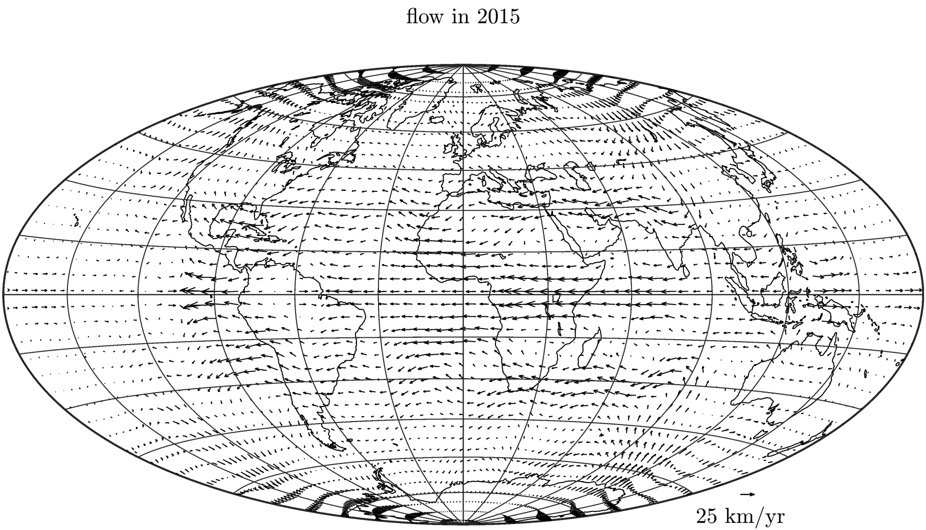
- Compare solutions with different flow regularization norms $\mathbf{R}(\mathbf{a})$:
  - L2 norm of flow horizontal divergence and radial vorticity *[e.g. Gillet et al., 2009]*

$$\int_{\mathrm{CMB}} (\nabla_{\mathrm{H}} \cdot \mathbf{u})^2 + (\mathbf{e}_r \cdot \nabla \times \mathbf{u})^2 \mathrm{d}S$$

  - L1 norm of mode amplitudes, implemented iteratively *[Farquharson and Oldenburg,1998]*

$$|\mathbf{a}|_1 = \sum_K |a_G^K| + \sum_M \sum_N |a_I^{M,N}|$$

Results I: L2 norm inversion



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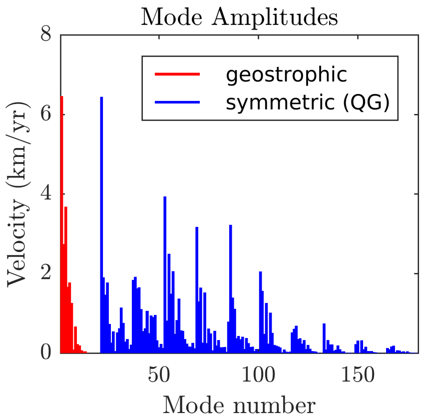

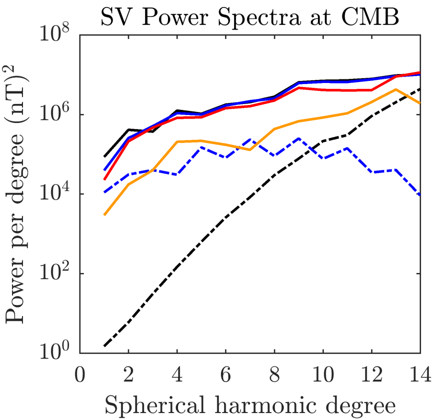

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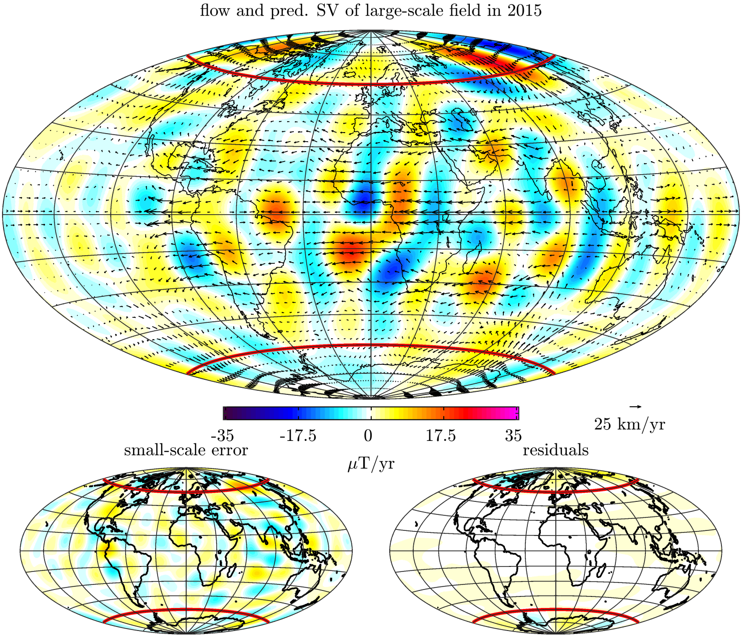
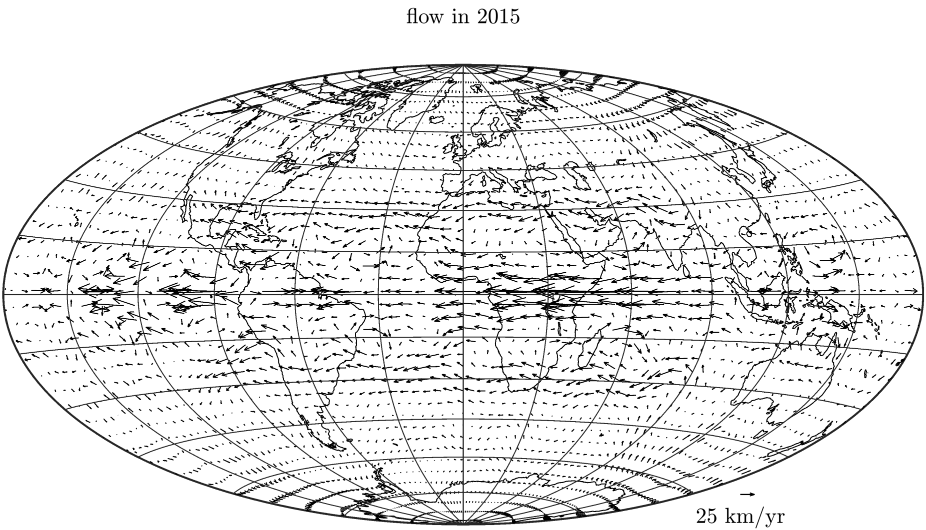


Results II: L1 norm inversion



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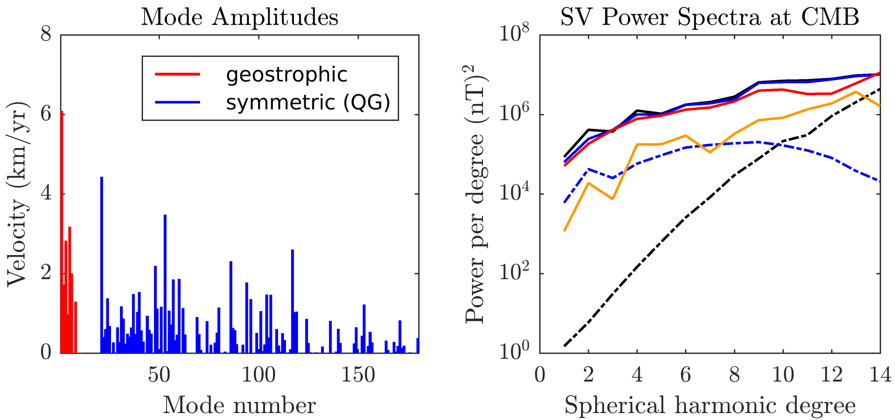

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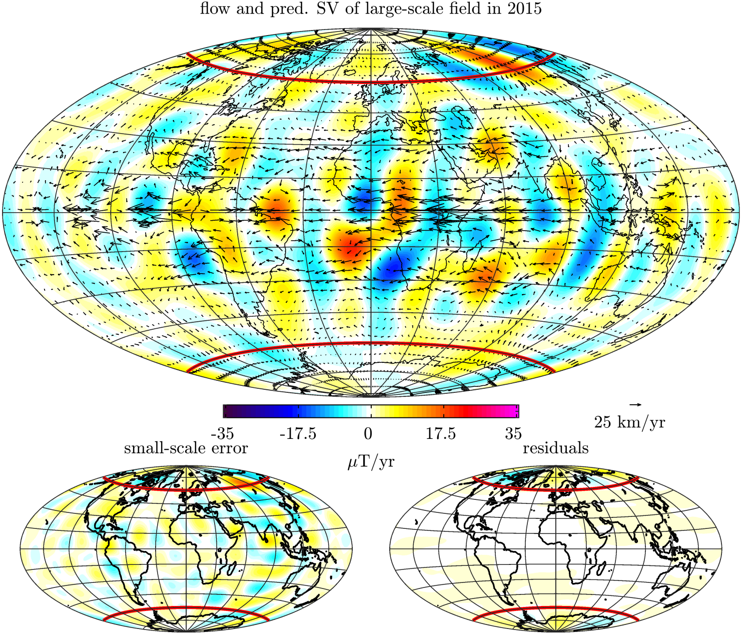
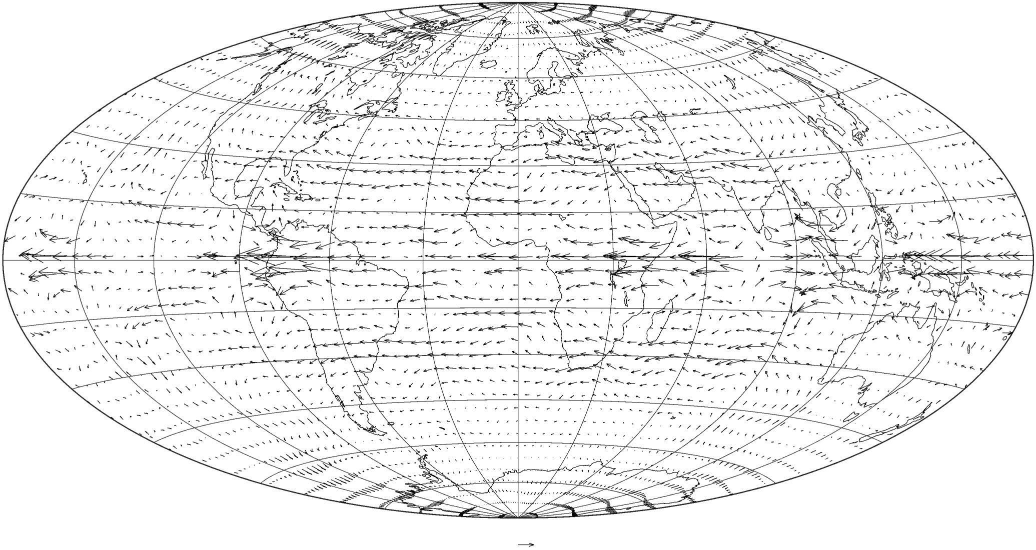


Results III: Time-dependence of L1 solution

Flow in 1999.0



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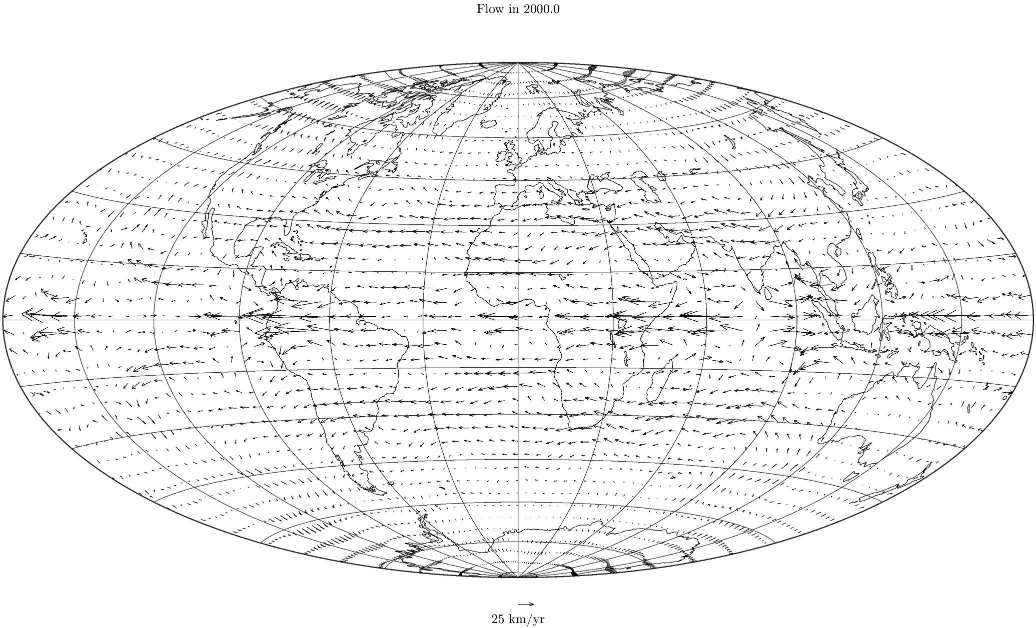

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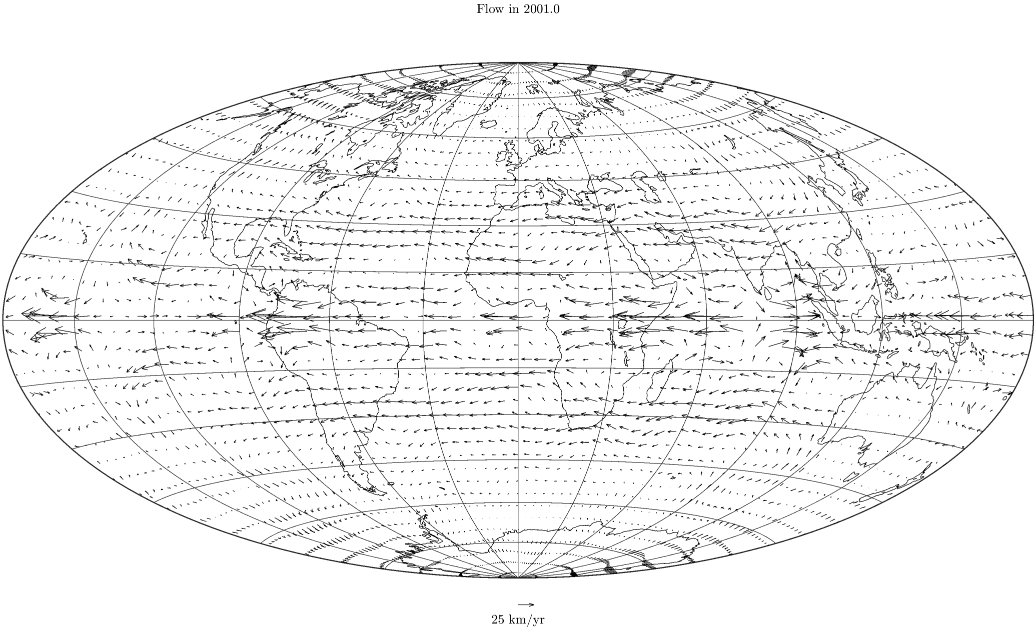

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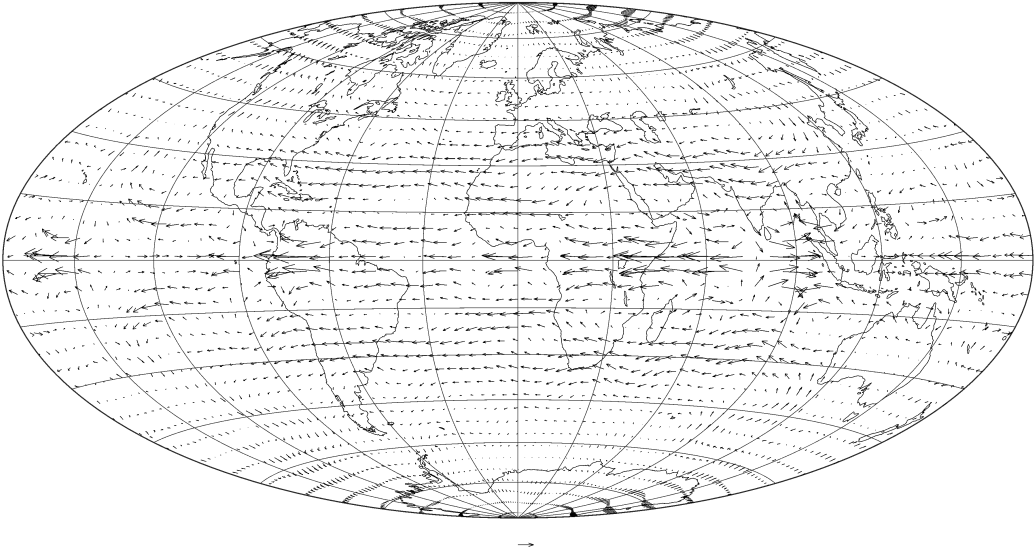
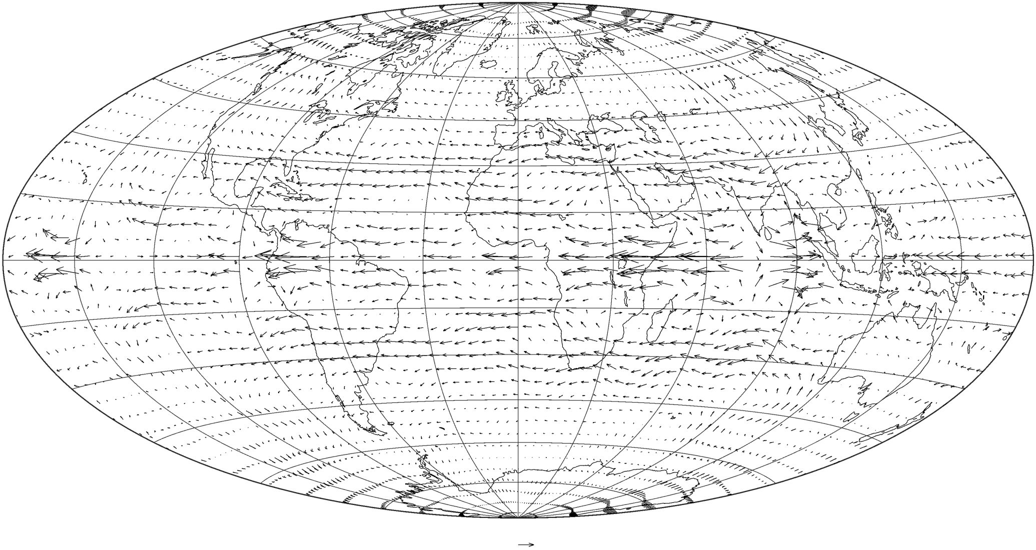
Flow in 2002.0



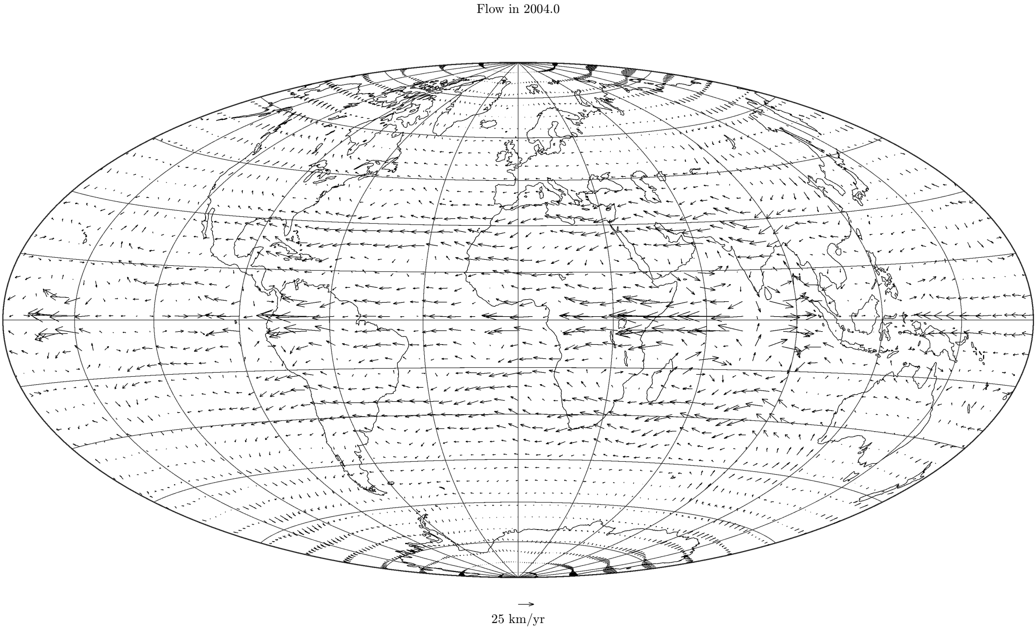
25 km/yr

Results III: Time-dependence of L1 solution

Flow in 2003.0



Results III: Time-dependence of L1 solution



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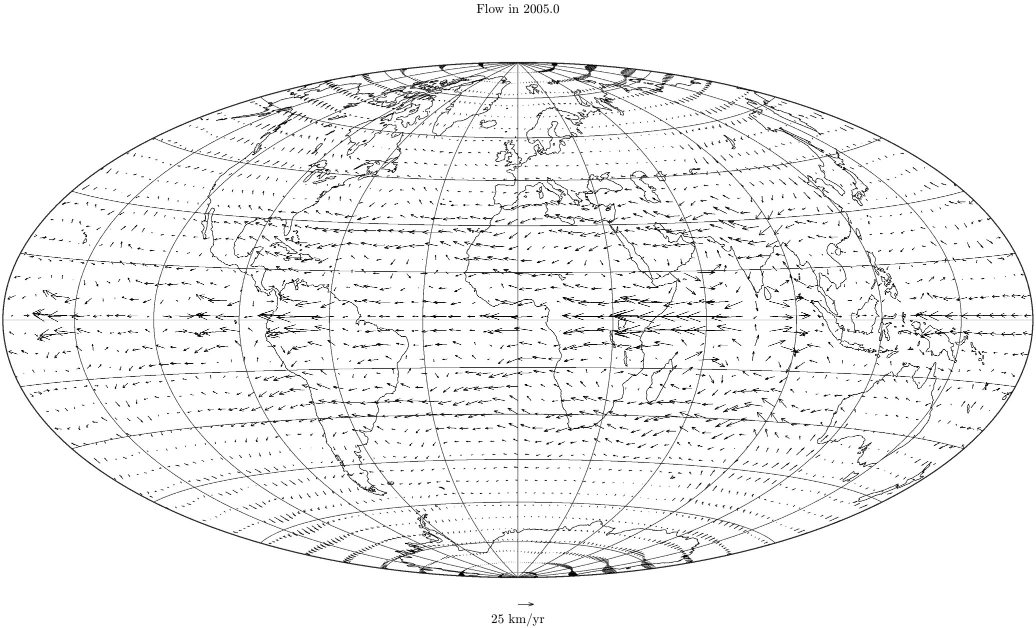

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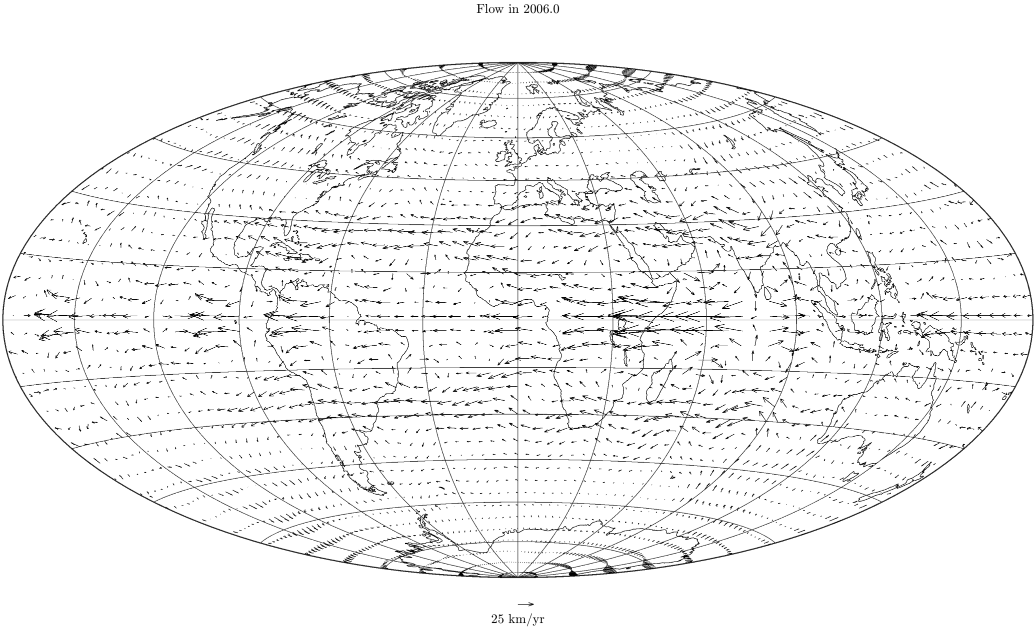
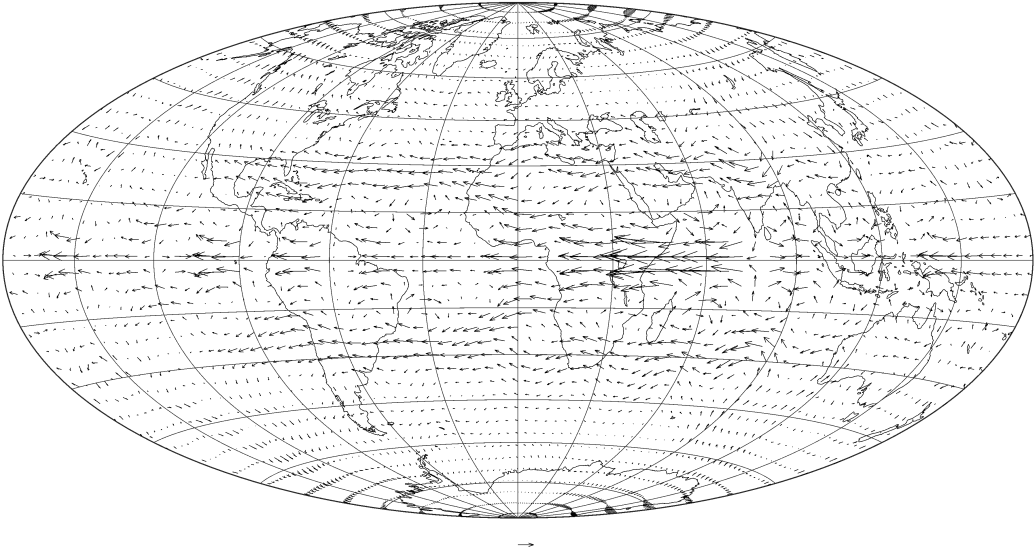


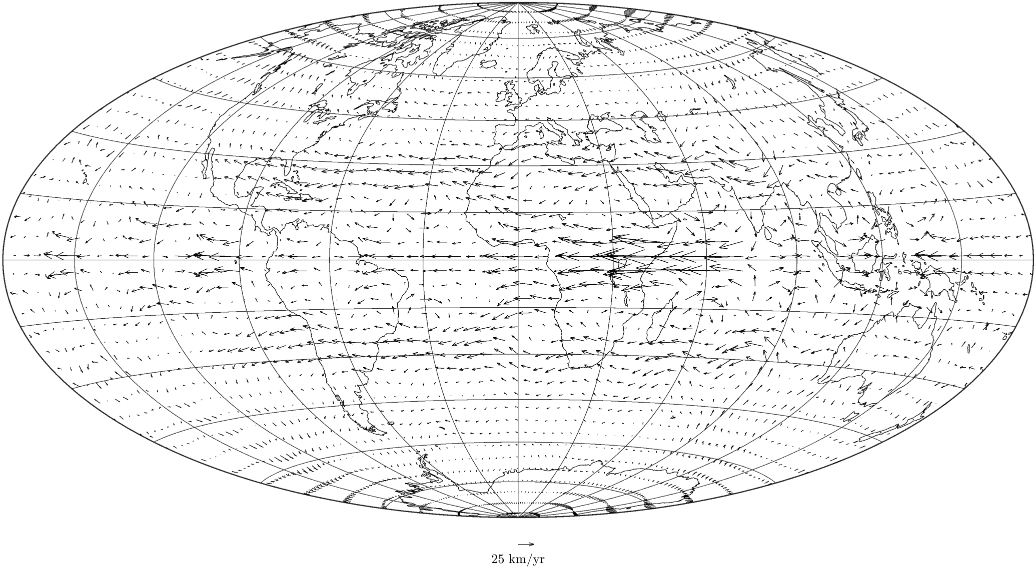
Results III: Time-dependence of L1 solution

Flow in 2007.0



Results III: Time-dependence of L1 solution

Flow in 2008.0



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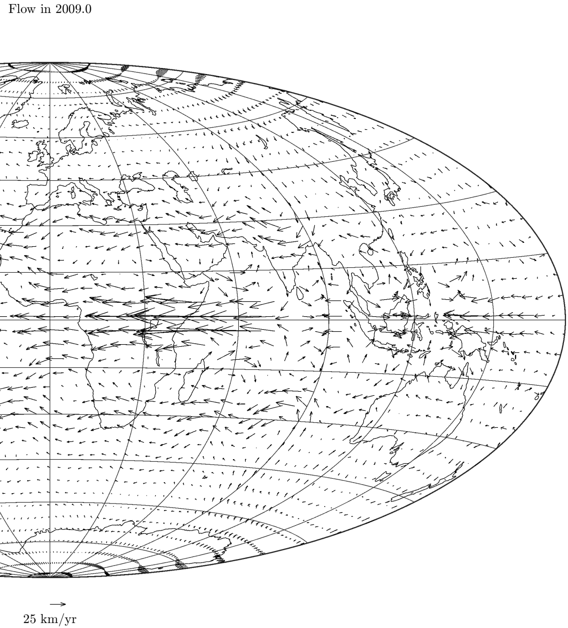
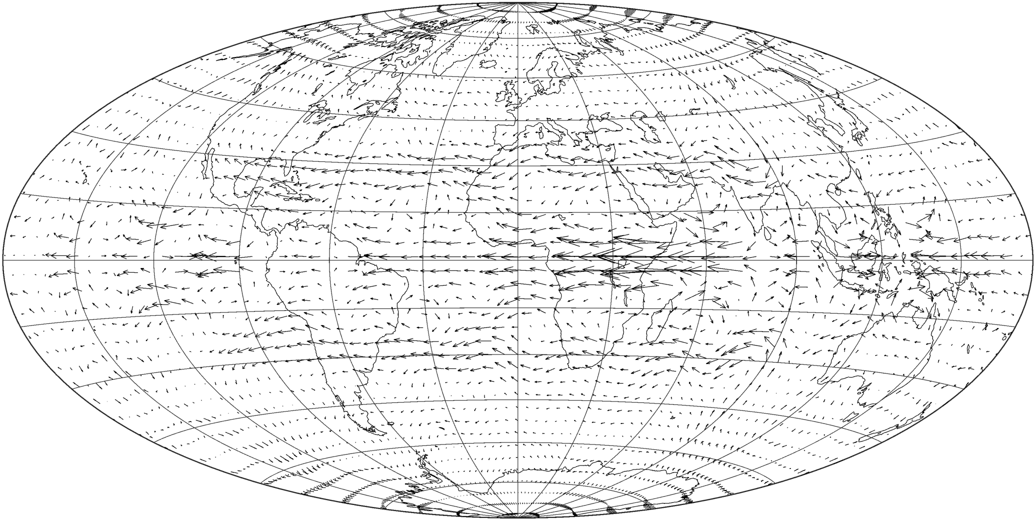


Results III: Time-dependence of L1 solution

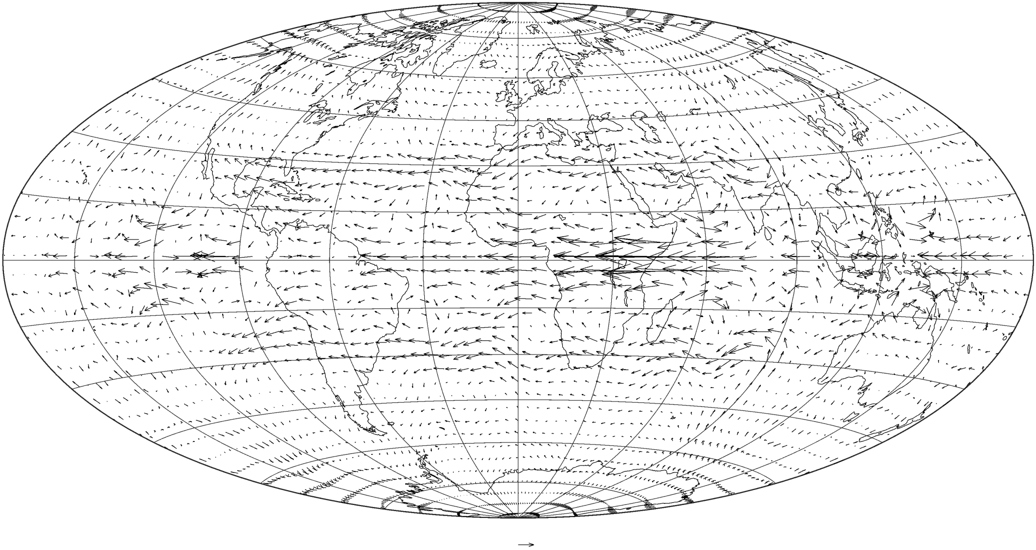
Flow in 2010.0



25 km/yr

Results III: Time-dependence of L1 solution

Flow in 2011.0



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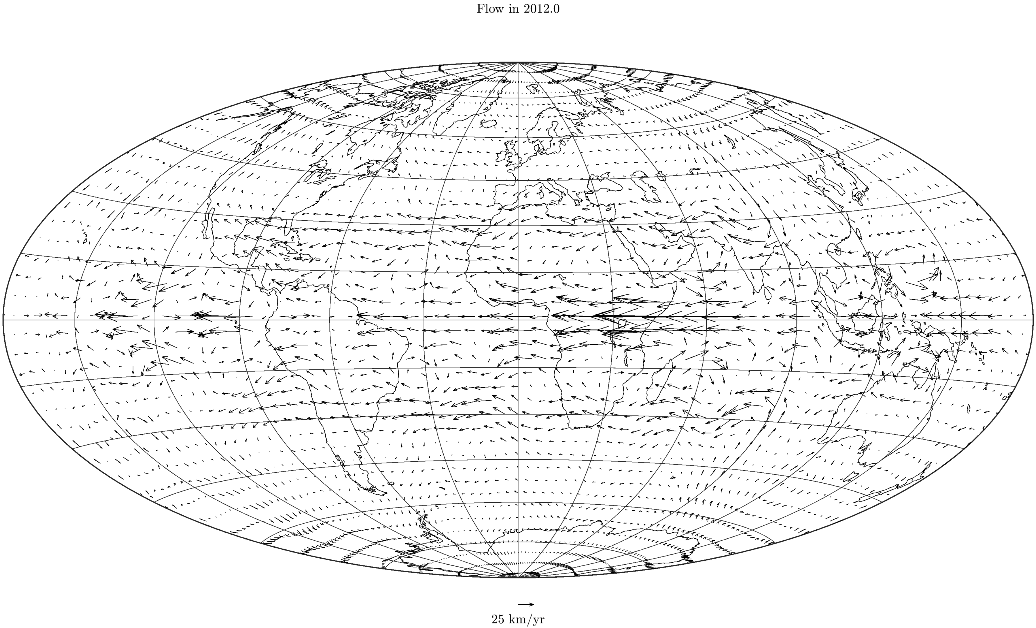

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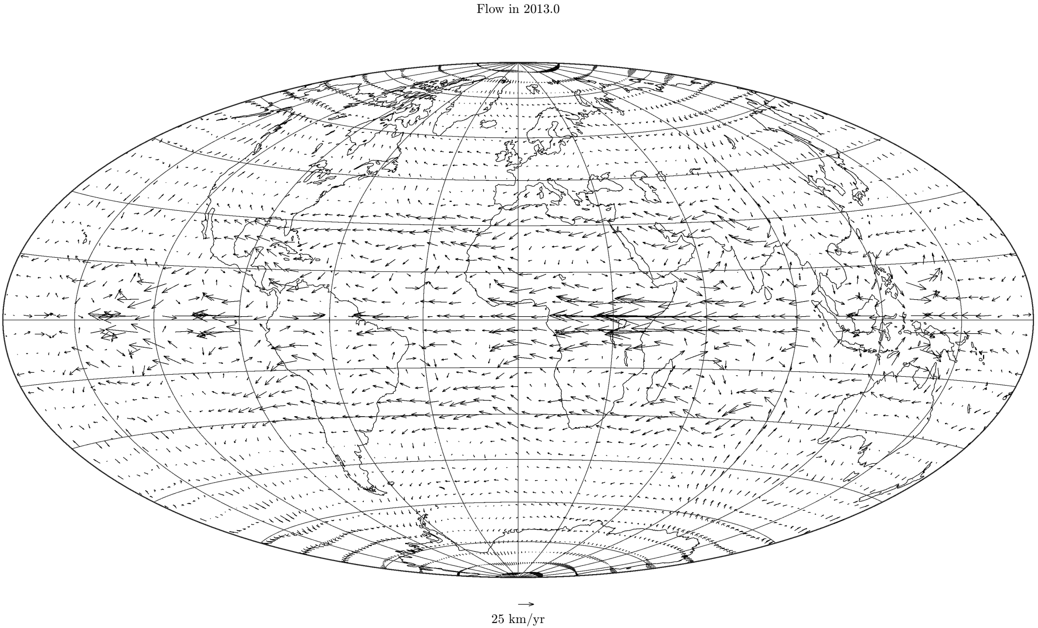

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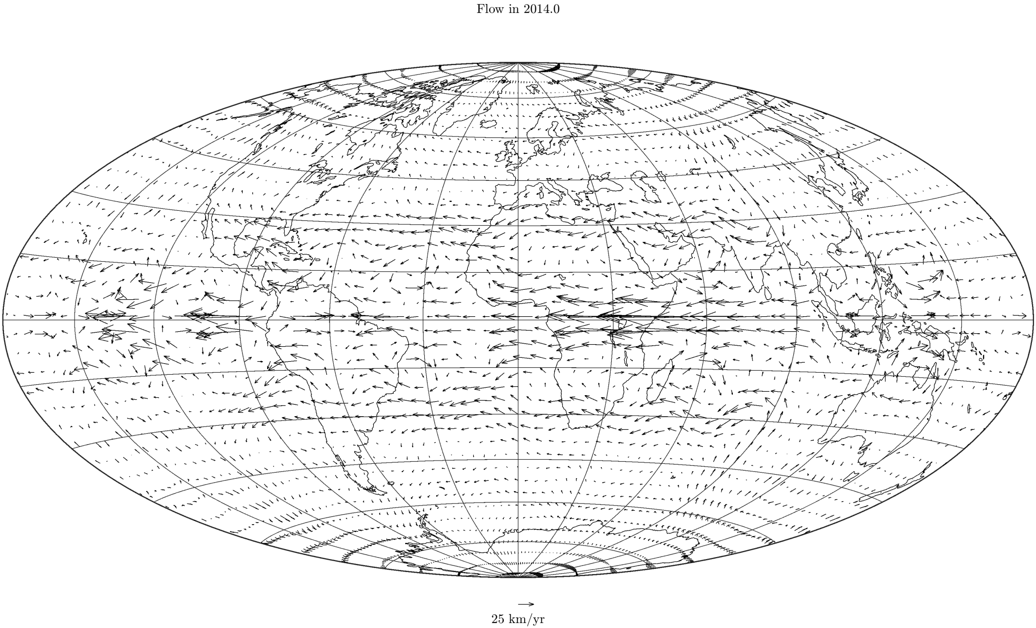
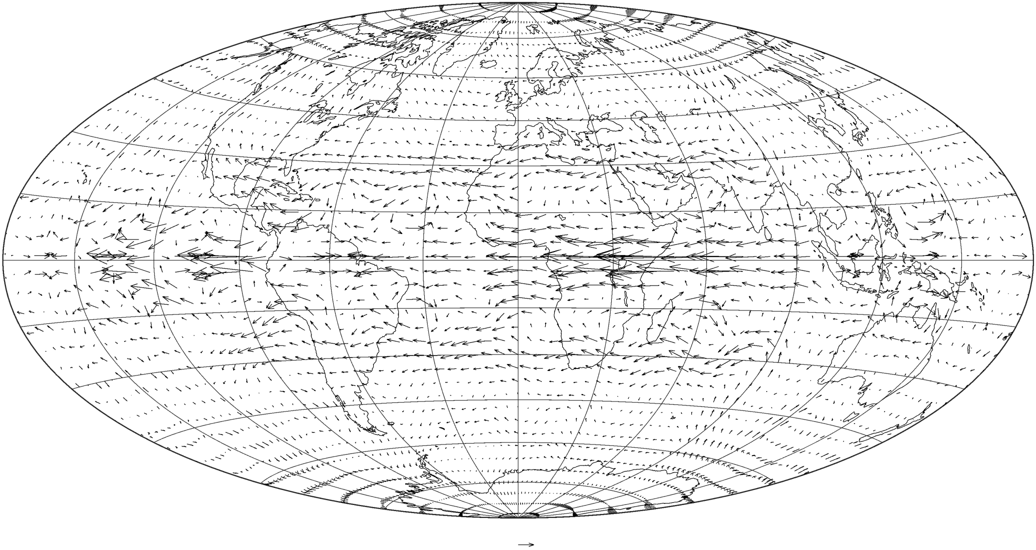


Results III: Time-dependence of L1 solution

Flow in 2015.0



25 km/yr

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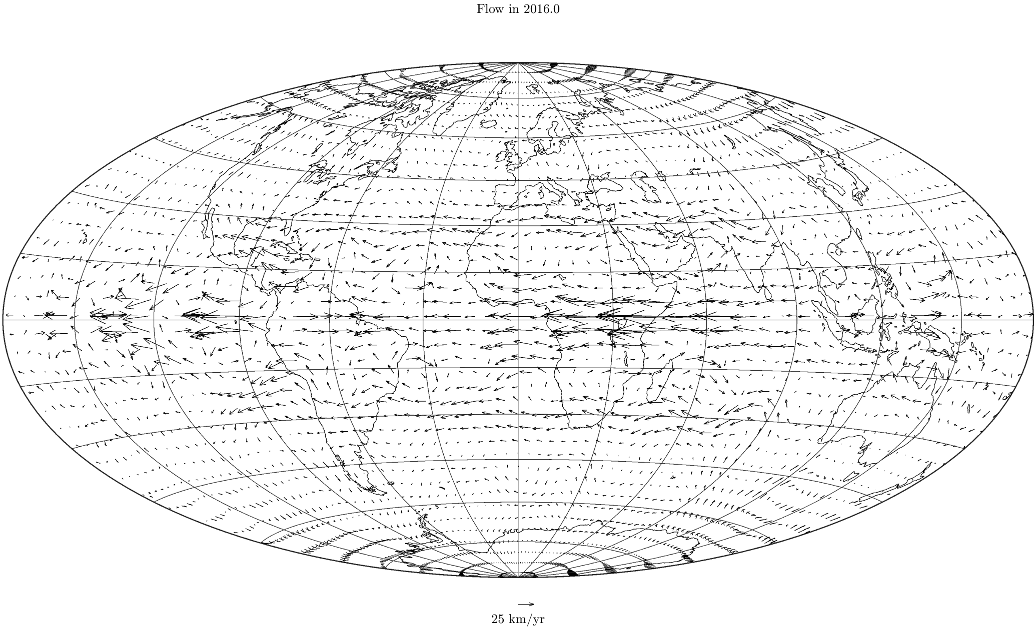

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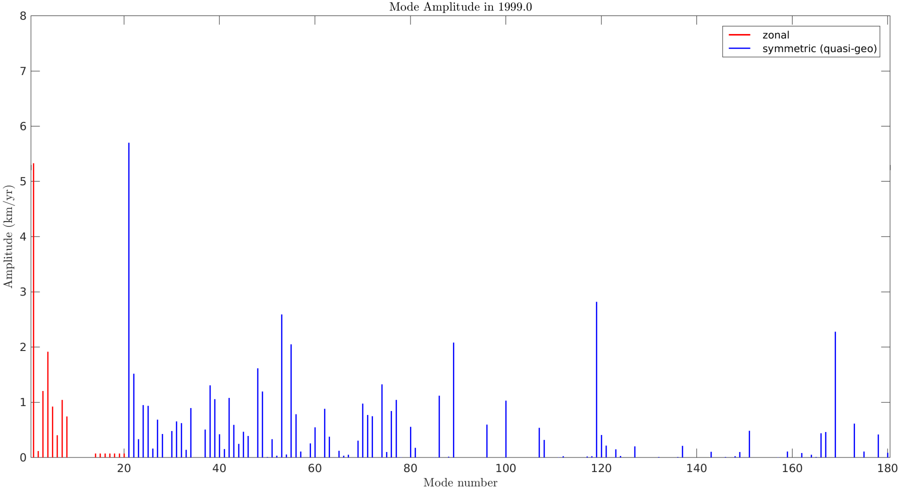

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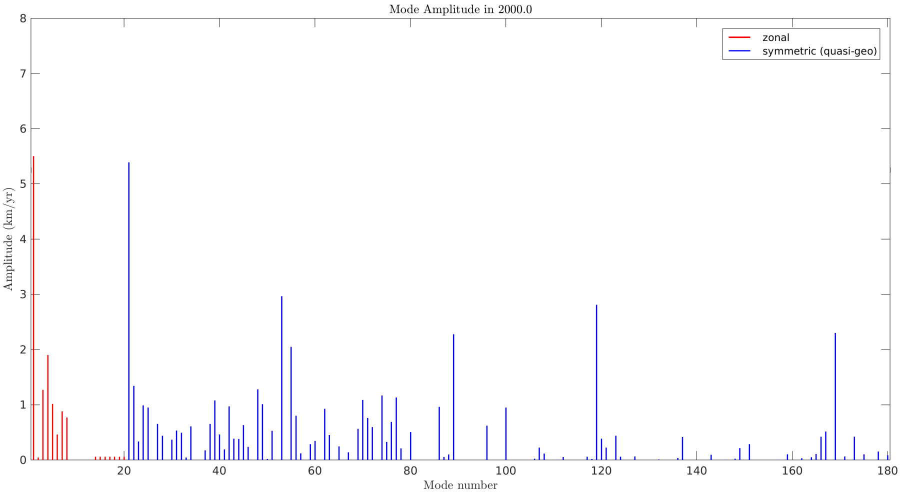

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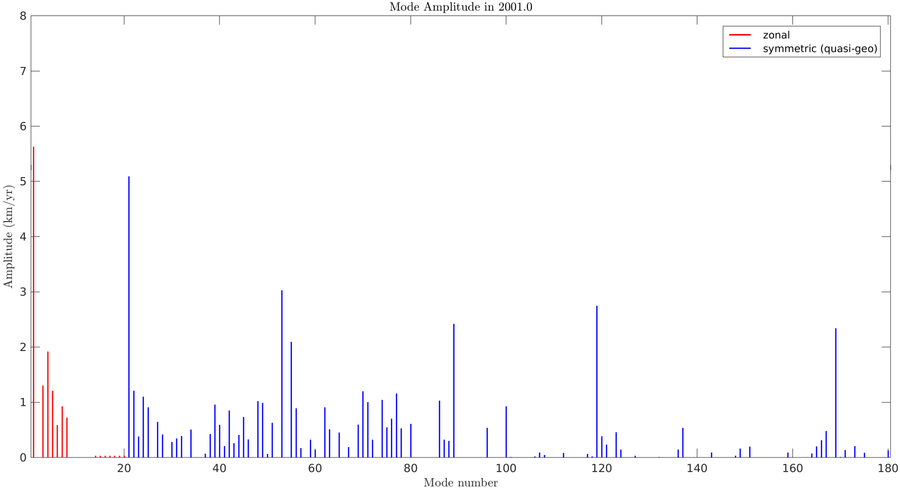

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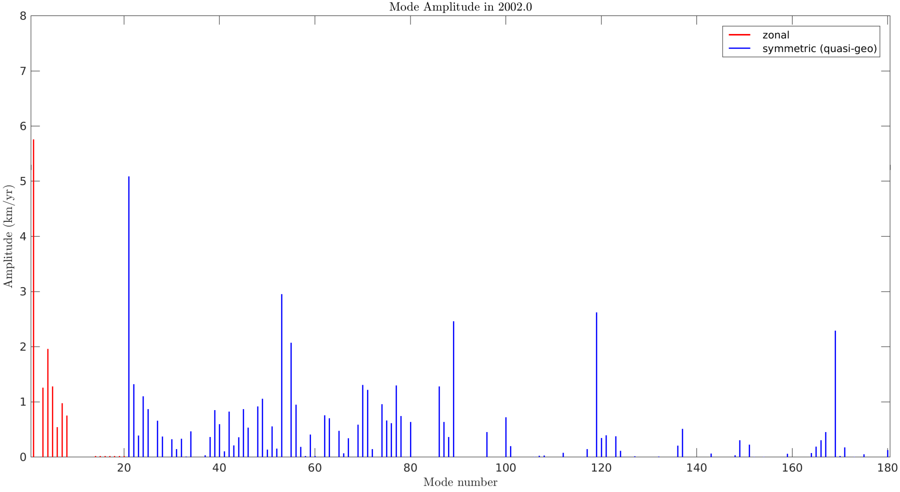

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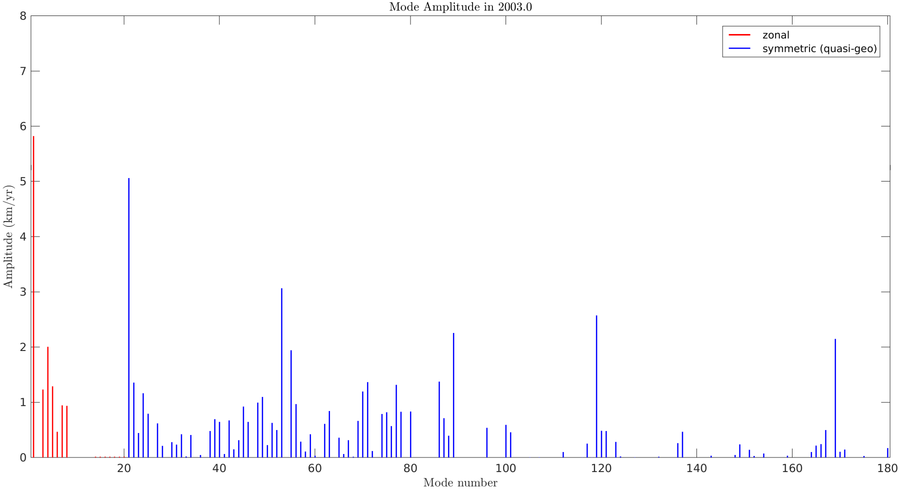

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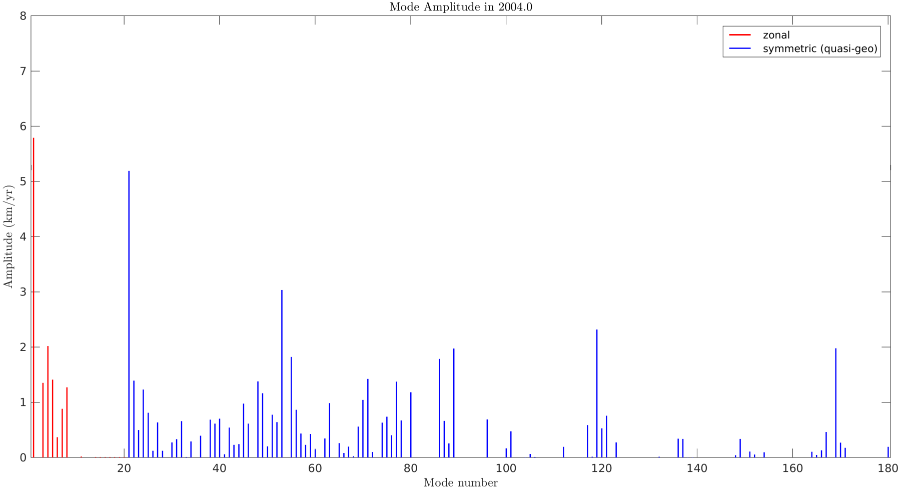

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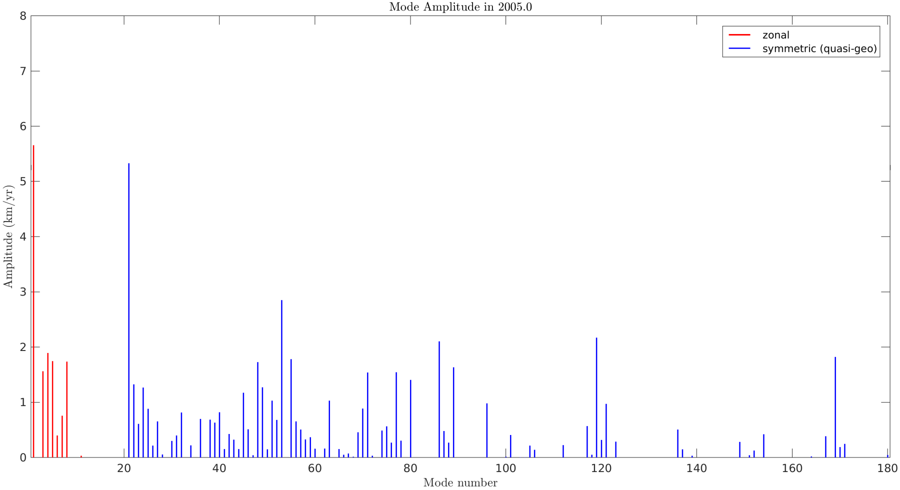

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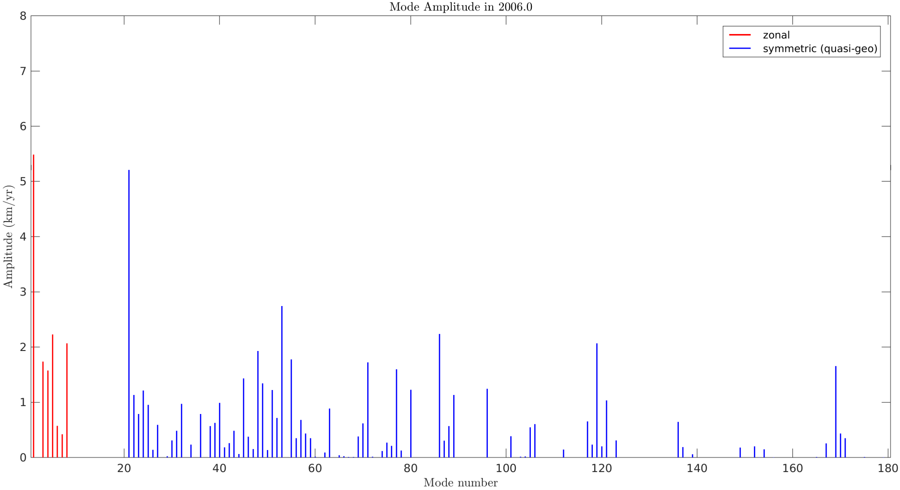

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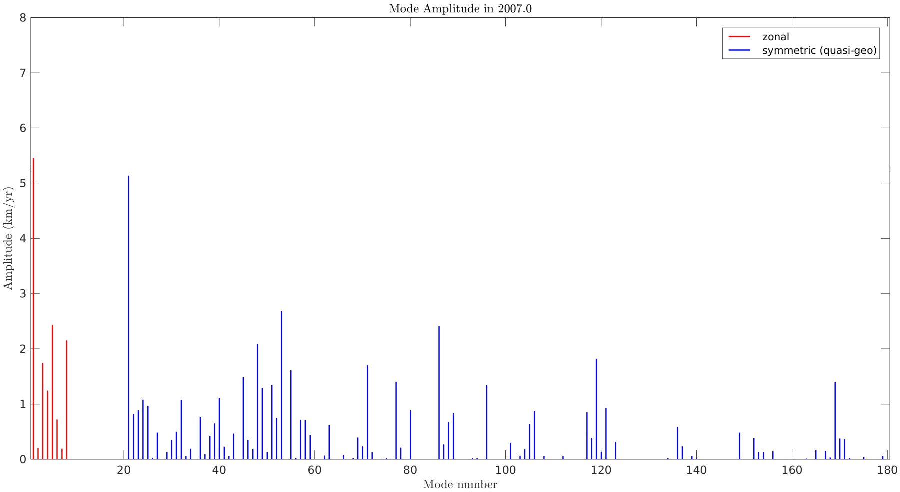

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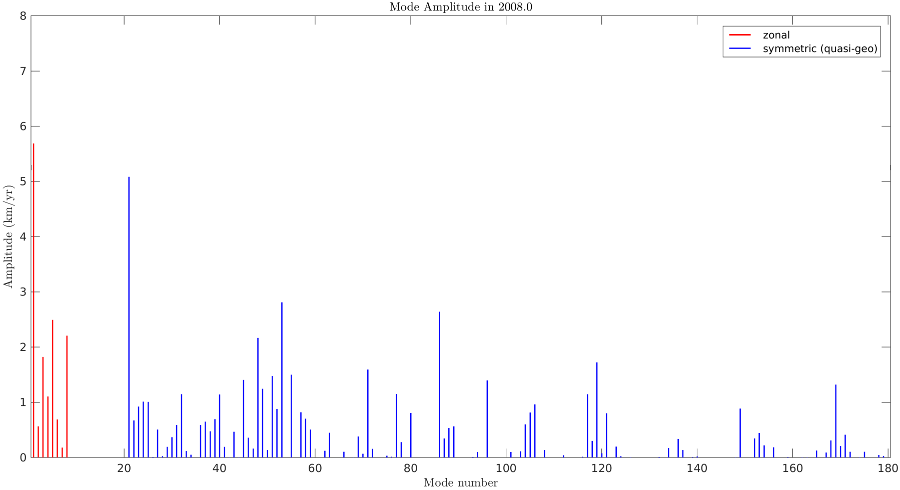

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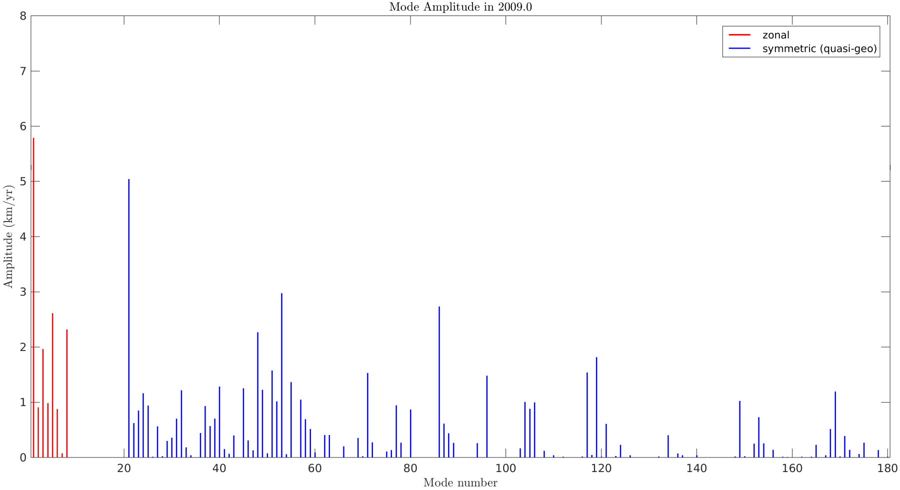

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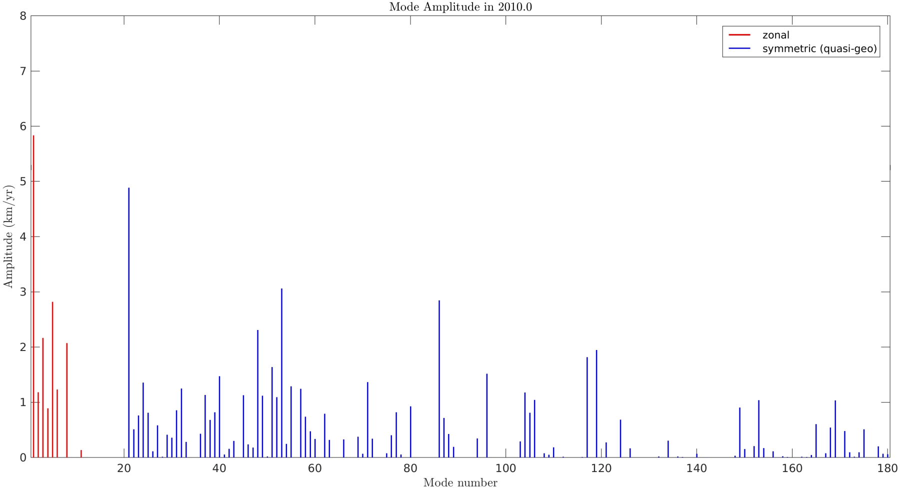

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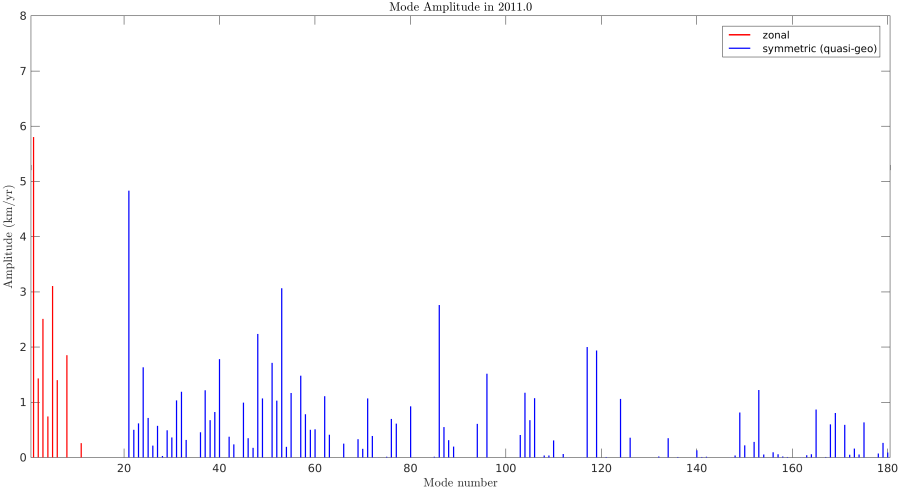

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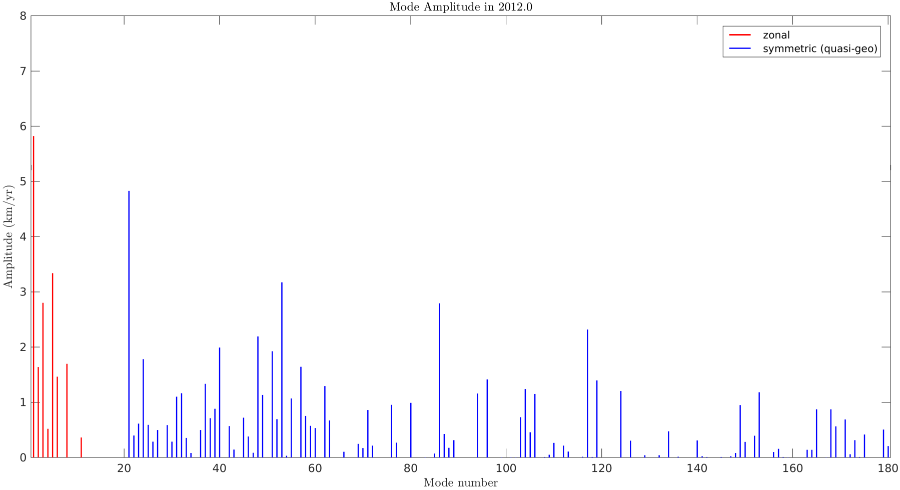

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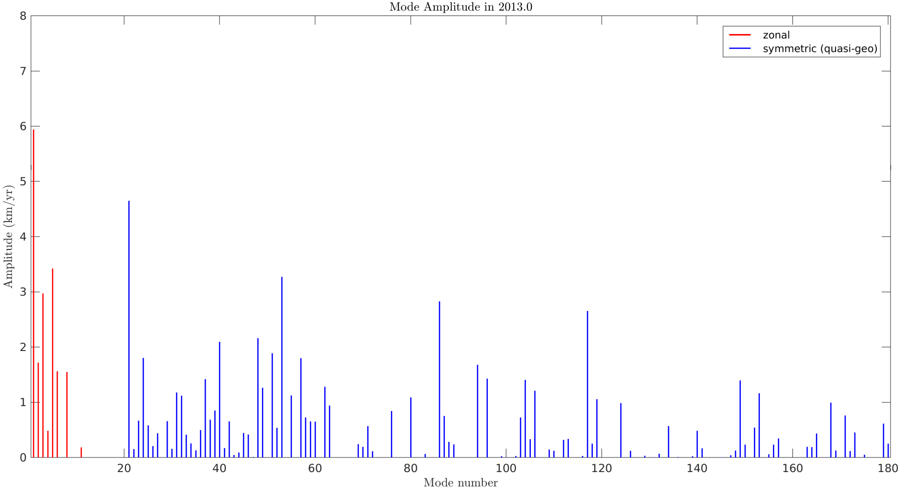

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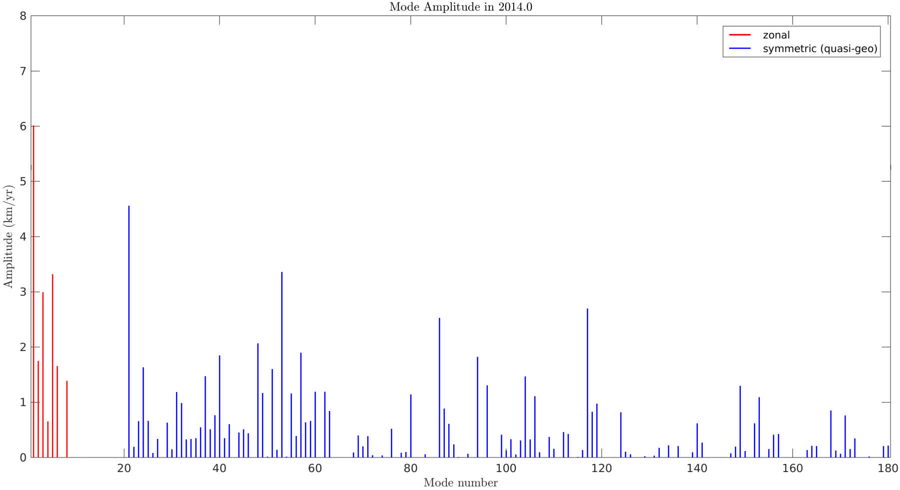
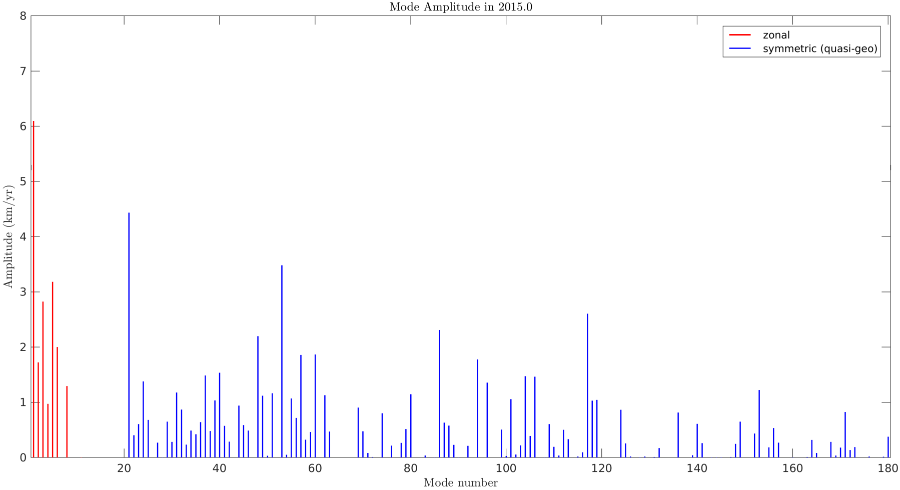
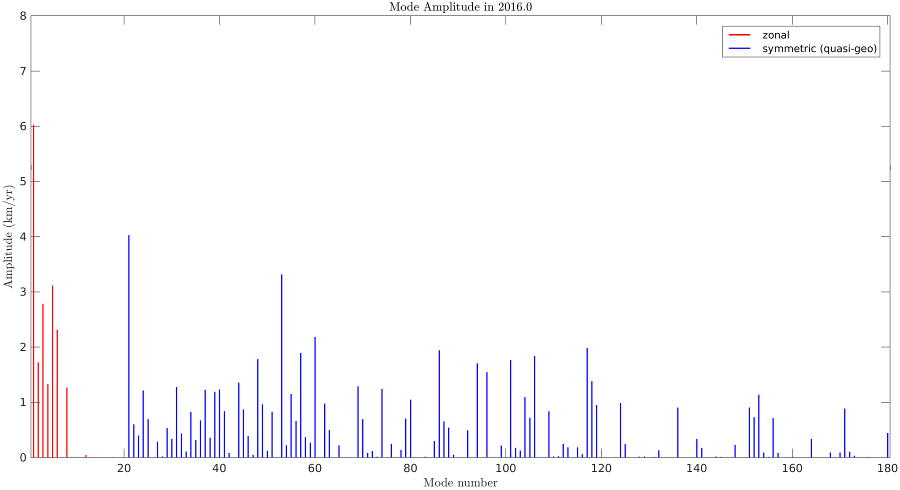


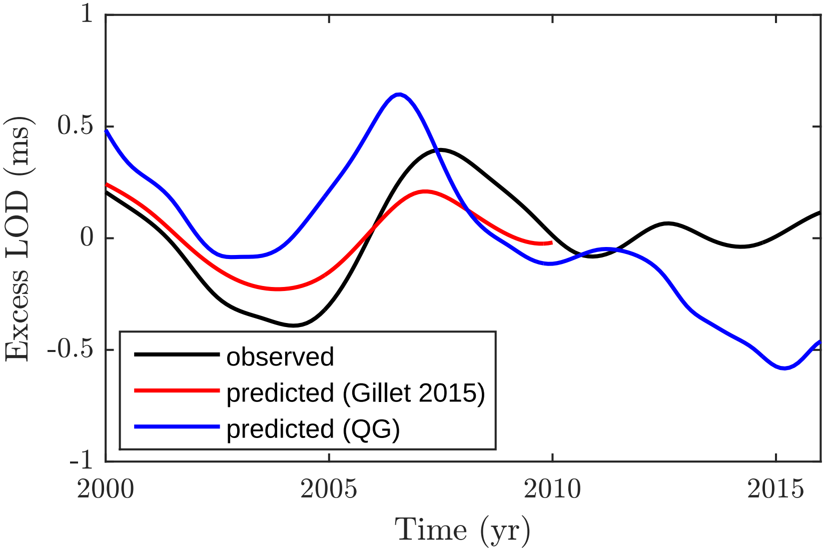
Results III: Time-dependence of L1 solution



Results III: Time-dependence of L1 solution

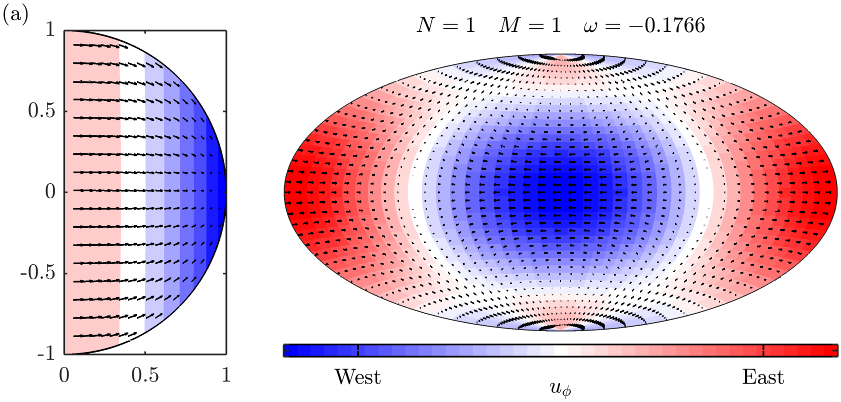


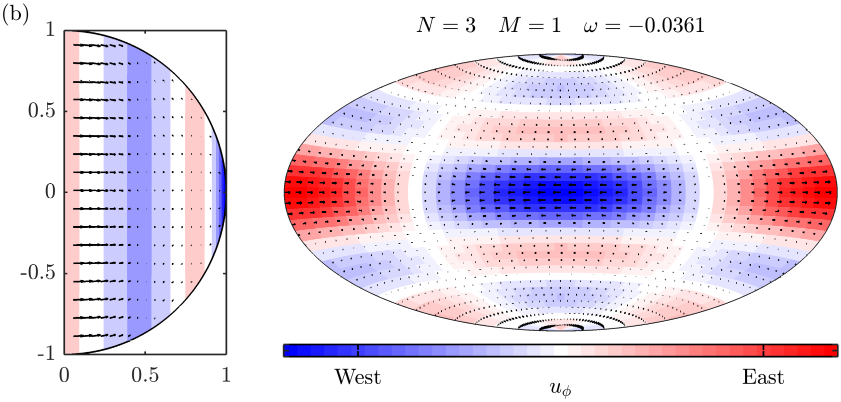
Results III: Time-dependence of L1 solution



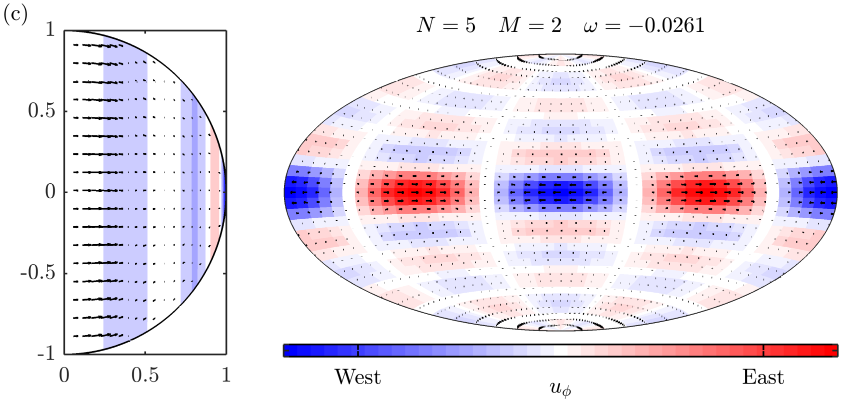
Discussion

- Basis of geostrophic and quasi-geostrophic modes is an alternative means of parameterizing core motions responsible for SV, suitable for studying equatorial region
- Allowing for impact of unresolved scales, can adequately fit observed CMB SV
- Penalizing L1 norm of mode amplitudes, find certain modes dominate (esp. $M = 1$) and some exhibit rapid time variations
- Approach can be extended to study more general flows using basis of E^S and E^A inertial modes
- Extensions
  - Solve for background flow (E^S & E^A) plus time-dependent QG/Geost. modes *[see also poster by N. Gillet]*
  - Directly fit satellite and ground data rather than field model
  - Solve for time-dependent mode amplitudes, link to dynamics *[Labbé et al., 2015]*
- Comparisons with more complete forward studies of low latitude core dynamics needed

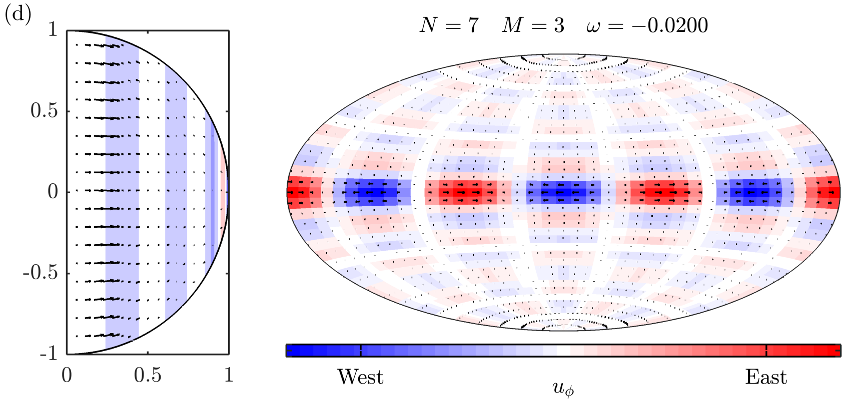

(a)
1
0.5
0
-0.5
-1
0
0.5
1
$N = 1 \quad M = 1 \quad \omega = -0.1766$
West
u_ϕ
East



(b)
1
0.5
0
-0.5
-1
0
0.5
1
$N = 3 \quad M = 1 \quad \omega = -0.0361$
West
u_ϕ
East

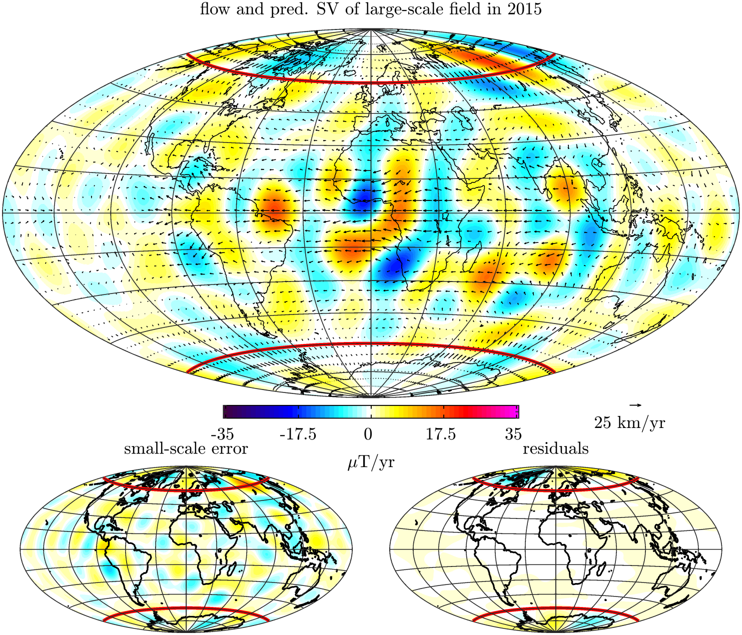


(c)
1
0.5
0
-0.5
-1
0
0.5
1
$N = 5 \quad M = 2 \quad \omega = -0.0261$
West
u_ϕ
East

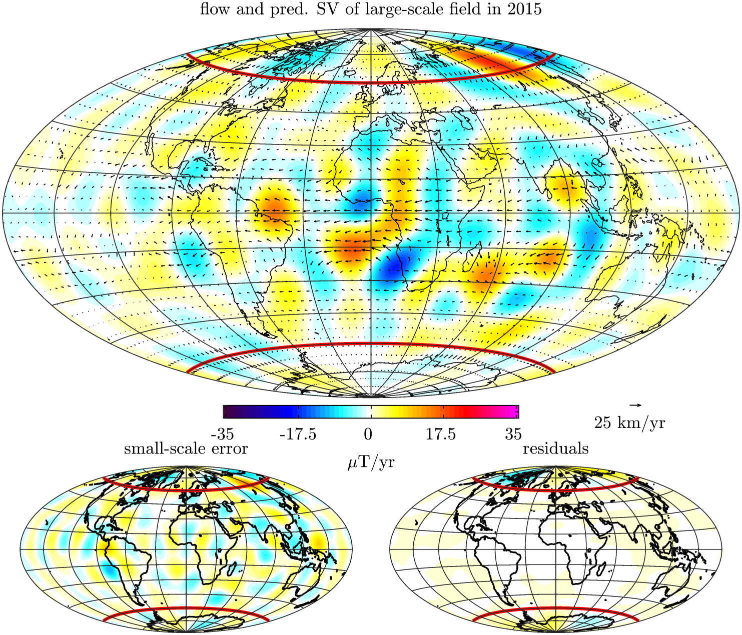


(d)
1
0.5
0
-0.5
-1
0
0.5
1
$N = 7 \quad M = 3 \quad \omega = -0.0200$
West
u_ϕ
East

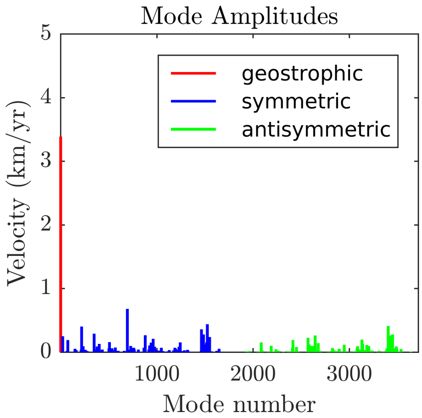
Geostrophic & E^S modes, L1 norm

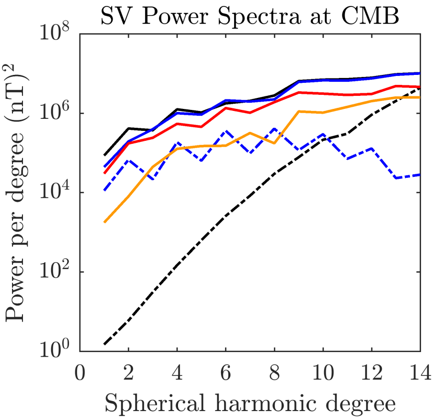


Geostrophic, E^S modes and E^A modes, L1 norm

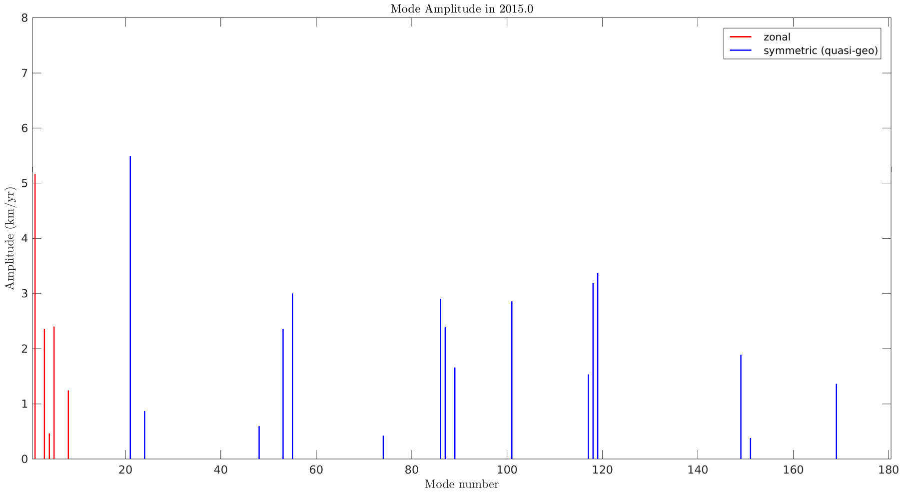


Geostrophic, E^S modes and E^A modes, L1 norm

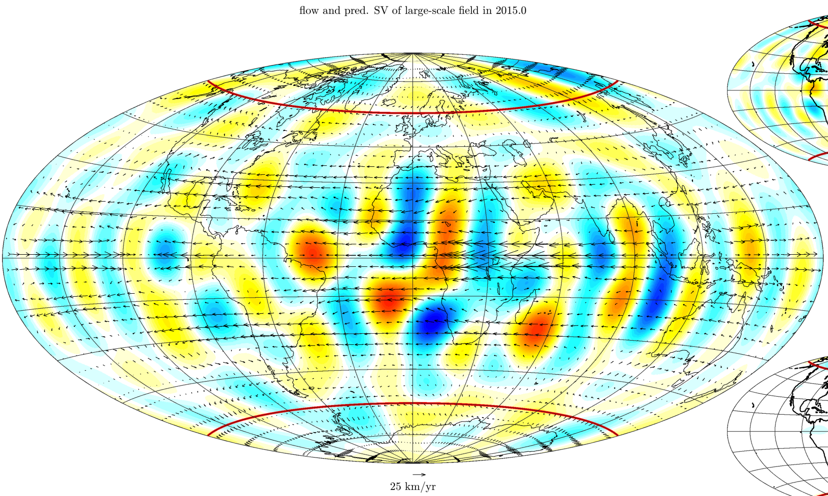


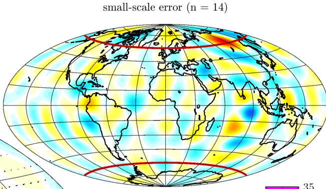


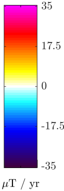
Restricted Geostrophic & QG basis

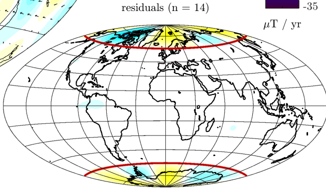


Restricted Geostophic & QG basis









E^S inertial mode solutions

$$u_r = -\mathrm{i} \sum_{i=0}^{N} \sum_{j=0}^{N-i} C_{ij;NM} r^{M+2(i+j)-1} \sin^{M+2j} \theta \cos^{2i} \theta \mathrm{e}^{\mathrm{i}M\phi} \cdot \sigma^{2i-1}(1-\sigma^2)^{j-1}[\sigma(M + M\sigma + 2j\sigma) - 2i(1-\sigma^2)] \tag{7a}$$

$$u_\theta = -\mathrm{i} \sum_{i=0}^{N} \sum_{j=0}^{N-i} C_{ij;NM} r^{M+2(i+j)-1} \sin^{M+2j-1} \theta \cos^{2i-1} \theta \mathrm{e}^{\mathrm{i}M\phi} \cdot \sigma^{2i-1}(1-\sigma^2)^{j-1}[\sigma(M + M\sigma + 2j\sigma)\cos^2\theta + 2i(1-\sigma^2)\sin^2\theta] \tag{7b}$$

$$u_\phi = \sum_{i=0}^{N} \sum_{j=0}^{N-i} C_{ij;NM} r^{M+2(i+j)-1} \sin^{M+2j-1} \theta \cos^{2i} \theta \mathrm{e}^{\mathrm{i}M\phi} \cdot \sigma^{2i}(1-\sigma^2)^{j-1}(M + M\sigma + 2j), \tag{7c}$$

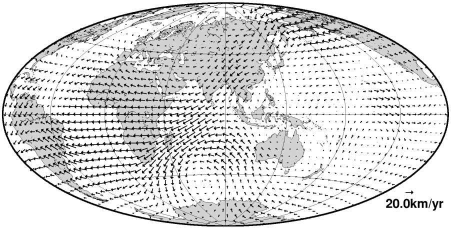
where the coefficients $C_{ij;NM}$ are defined as

$$C_{ij;NM} = \frac{(-1)^{i+j}[2(N+M+i+j)-1]!!}{2^{j+1}(2i-1)!!(N-i-j)!i!j!(M+j)!}. \tag{8}$$

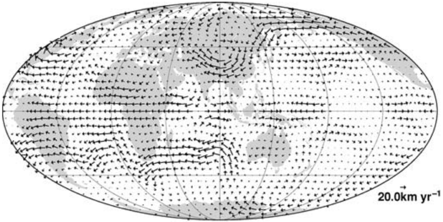
and the half-frequencies, σ are the roots of the polynomial

$$0 = \sum_{j=0}^{N} (-1)^j \frac{[2(2N+M-j)]!}{j!(2N+M-j)![2(N-j)]!} \left[(M+2N-2j) - \frac{2(N-j)}{\sigma} \right] \sigma^{2(N-j)} \tag{9}$$

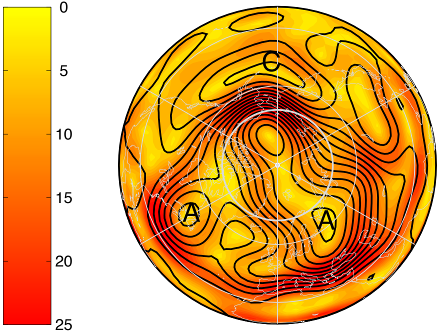
Comparisons with previous flow inversions



Whaler et al. (2016)



Holme and Olsen (2006)



Gillet et al. (2015)

The CHAOS-6 geomagnetic field model

- CHAOS series of geomagnetic field models aims to describe the near-Earth magnetic field to high spatial and temporal resolution *(Olsen et al., 2006, 2009, 2010, 2014)*

- **Potential field approach:** $\mathbf{B} = -\nabla V$ where $V = V^{\mathrm{int}} + V^{\mathrm{ext}}$.

- The internal part of the potential takes the form

$$V^{\mathrm{int}} = a \sum_{n=1}^{N_{\mathrm{int}}} \sum_{m=0}^{n} \left(g_n^m \cos m\phi + h_n^m \sin m\phi \right) \left(\frac{a}{r} \right)^{n+1} P_n^m \left(\cos\theta \right)$$

- For $n \leq 20$, expand in 6th order B-splines

$$g_n^m(t) = \sum_{k=1}^{K} {}^k g_n^m B_k(t).$$

- Also co-estimate the large-scale magnetospheric field

- And work with satellite vector data in **magnetometer frame**, co-estimating Euler angles

CHAOS-6 model: Parameterization of the external Field

- For the external potential, expand in SM and GSM co-ordinate systems, with θ_d and T_d being dipole co-lat. and dipole local time

$$
\begin{aligned}
V^{\mathrm{ext}} &= a \sum_{n=1}^{2} \sum_{m=0}^{n} \left(q_n^m \cos mT_d + s_n^m \sin mT_d\right) \left(\frac{r}{a}\right)^n P_n^m(\cos\theta_d) \\
&+ a \sum_{n=1}^{2} q_n^{0,\mathrm{GSM}} R_n^0(r,\theta,\phi).
\end{aligned}
$$

- Degree-1 coefficients in *SM* coords dependent on the RC disturbance index

CHAOS-6 model

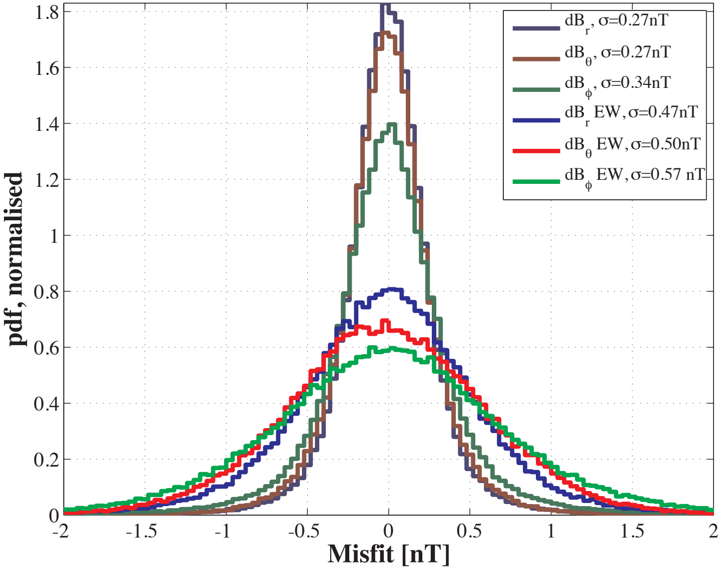
- **Use DTU's latest geomagnetic model**, CHAOS-6 *(Finlay et al., 2016)*

  `http://www.spacecenter.dk/files/magnetic-models/CHAOS-6/`

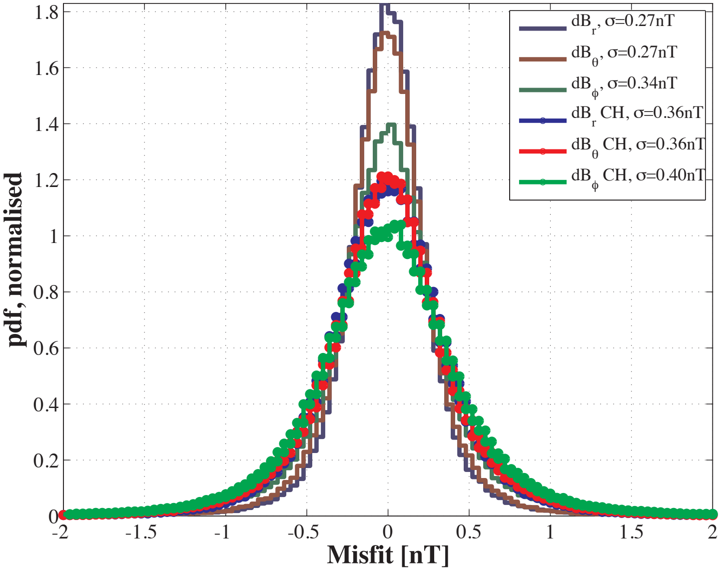
- Derived from 7,873,156 data

- Weighted rms misfit to non-polar, dark *Swarm* scalar data is **2.14 nT**,
  For scalar field differences, **0.26 nT** along-track and **0.45 nT** across-track.

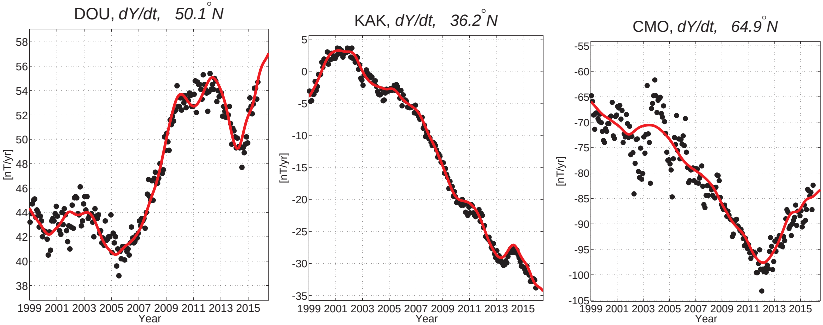
Fit to Swarm field difference data: histograms of residuals



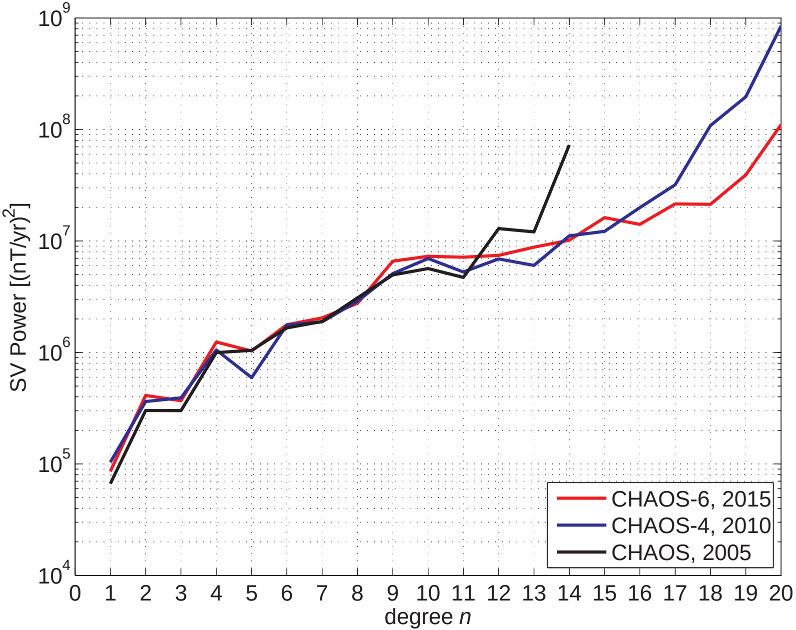
Vector difference residuals, Swarm vs CHAMP



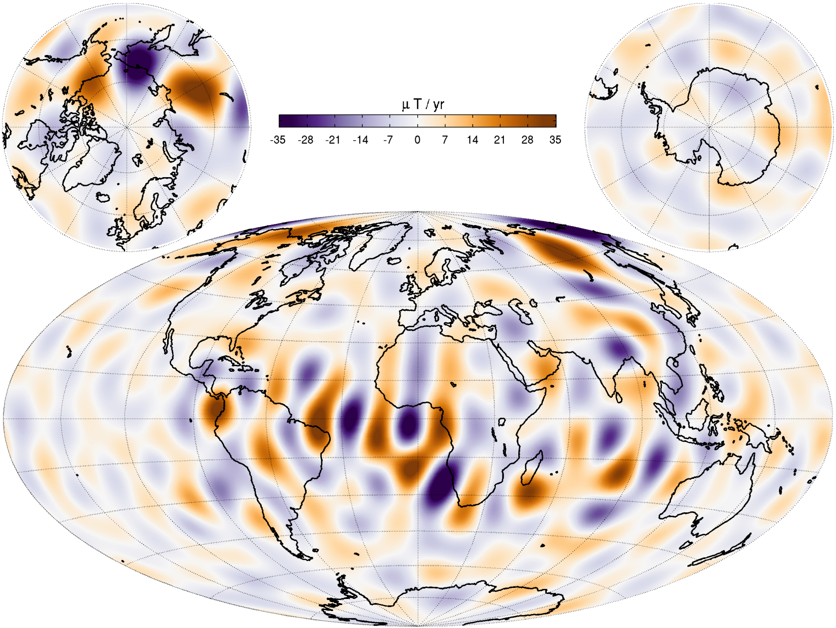
Fit to ground observatory data, Eastward component dY/dt



Power spectrum of SV at core surface



Time-dependence of core surface SV



Time-dependence of core surface SA